

DESIGN CRITERIA FOR BUCKLING AND VIBRATION OF IMPERFECT STIFFENED SHELLS\*

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Abstract

Recent advances in the methods of prediction of buckling loads of stiffened shells and their applicability to design are reviewed. The influence of boundary conditions, imperfections and inelastic effects are studied and correlated with tests. Realistic design criteria are evolved.

Methods for vibration analysis of loaded stiffened shells are evaluated by correlation with test results and consideration of design application. The influence of boundary conditions and imperfections are studied and correlated with their influence on instability.

The use of vibration testing as a non-destructive method for checking of actual boundary conditions, and for prediction of buckling loads of stiffened shells is discussed and correlated to tests.

List of Symbols

$A_1, A_2$  cross section of area of stringer and ring, respectively.  
 $b_1, b_2$  distance between centers of stringers and rings, respectively.  
 C1:  $w=w, x=N=N, \phi=0$   
 C2:  $w=w, x=u=N, \phi=0$   
 C3:  $w=w, x=N=N, \phi=v$   
 C4:  $w=w, x=u=N, \phi=v$   
 $c_1, c_2$  stringer and ring width, respectively  
 $D$   $Eh^3/12(1-\nu^2)$   
 $d_1, d_2$  stringer and ring height, respectively  
 $E$  Young's modulus of shell  
 $E_1, E_2$  Young's modulus of stringer and ring, respectively  
 $e_1, e_2$  stringer or ring eccentricity, respectively  
 $\bar{e}$  eccentricity of loading (distance from shell middle surface to the point of application of load)  
 $f$  frequency  
 $G$  shear modulus of shell  
 $G_1, G_2$  shear modulus of stringer and ring, respectively  
 $h$  thickness of shell  
 $\bar{h}$  thickness of equivalent weight isotropic shell  
 $I_{11}, I_{22}$  moment of inertia of stringer or ring cross-section about its centroidal axis

$I_{01}, I_{02}$  moment of inertia of stringer or ring cross-section about the middle surface of shell  
 $J_1, J_2$  torsion constant of stringer or ring cross-section, respectively  
 $k_1$  axial elastic restraint  
 $k_4$  rotational elastic restraint  
 $L$  length of shell

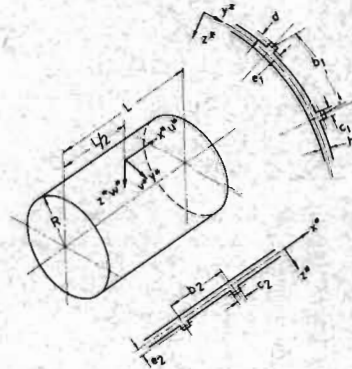


Figure 1. Notation.

$M$  mass per unit area of stiffened shell  
 $M_x, M_\phi, M_{x\phi}, M_{\phi x}$  moment resultants  
 $m$  number of longitudinal half waves  
 $N_x, N_\phi, N_{x\phi}$  axial, circumferential and shear membrane force resultants, respectively  
 $n$  number of circumferential waves  
 $P$  compressive axial load  
 $P_B$  empirical buckling load for isotropic cylindrical shell, Eq. (5).  
 $P_{cr}$  calculated axial buckling load  
 $P_{sp}^{exp}$  experimental axial buckling load  
 $P_{sp}$  calculated axial buckling load for shell with axial or rotational restraint  
 $P_{SS3}, P_{SS4}, P_{C4}$  calculated axial buckling loads for shell with SS3, SS4, C4 boundary conditions, respectively  
 $\bar{P}_{cl}$  classical buckling of equivalent weight isotropic shell  
 $P_{stiff}$  linear theory axial buckling load for stiffened cylindrical shell  
 $R$  radius to shell middle surface  
 SS1:  $w=M=N=N, \phi=0$   
 SS2:  $w=M^x=u^x=N^x, \phi=0$   
 SS3:  $w=M^x=N^x=v^x, \phi=0$   
 SS4:  $w=M^x=u^x=v^x, \phi=0$   
 $\Phi$   $h^2/12R^2$   
 $t^*$  time  
 $u, v, w$  displacements (see Fig. 1)  
 $u^*, v^*, w^*$  non-dimensional displacements  $u^*/R, v^*/R, w^*/R$  respectively  
 $x^*, y^*, z^*$  coordinates (see Fig.1)  
 $x, \phi, z$  non-dimensional coordinates  $x^*/R, y^*/R, z^*/R$  respectively  
 $Z$   $(1-\nu^2)^{1/2} (L/R)^2 (R/h)$ , Batdorf shell parameter

\* The research reported in this paper has been sponsored in part by the Air Force Office of Scientific Research, through the European Office of Aerospace Research, United States Air Force under Contract F44620-71-C-0116 and Grant 72-2394.

$\gamma$	ratio between the "linearities" of stiffened and equivalent weight isotropic shell
$\epsilon_x, \epsilon_\phi, \epsilon_{x\phi}$	normal and shear strains
$\eta$	efficiency of stiffened shell
$\mu$	nondimensional parameter which determines amplitude of imperfection
$\nu$	Poisson's ratio
$\rho$	"linearity", ratio between experimental buckling load and the theoretical value predicted by linear theory
$\rho_{SS3}, \rho_{SS4}, \rho_{C4}$	"linearity" with respect to SS3, SS4, C4 boundary conditions respectively
$\rho_{sp}$	"linearity" with respect to end conditions with axial or rotational restraint
$\rho_{eq}$	"linearity" or knock-down factor equivalent weight isotropic shell
$-2$	
$\rho^2$	$(1/\sqrt{3(1-\nu^2)})(h/R) m^2$
$\tau$	$(1/\sqrt{3(1-\nu^2)})(h/R) n^2$
$\mu_k$	$(1-\nu^2) E_{kA} / E_{kB} h$
$\chi_k$	$(1-\nu^2) E_{kA} e_k / E_{kB} h R$
$n_k$	$E_{kI} / b_k$
$n_{ok}$	$G_{kJ} / b_k$
$n_{tk}$	$G_{kI} / b_k$
$(\cdot)_{,r}$	differentiation by r

k is 1 for stringers and 2 for rings.

### I. Introduction

Stiffened shells are now well established as optimal configurations for major structural components of aerospace vehicles. Considerable theoretical and experimental effort has therefore been devoted in recent years towards better understanding of their most probable mode of failure - instability. In 1972, the senior author surveyed the state of the art of buckling analysis for integrally stiffened shells and the relevant experimental evidence<sup>1</sup>. One purpose of the present study is to update this survey and extend it to the vibrations of stiffened shells and the correlation of vibrations with instability. The emphasis is, however, on design applications, on evaluation of earlier design criteria and on formulation of new criteria for design of more efficient aerospace shells.

The following quotation from a recent paper by Fulton and McComb<sup>2</sup> may serve as a reminder of the primary importance of structural efficiency in future aerospace vehicles. "At the same time that design experience on vehicles classes is decreasing, the payload of advanced vehicles is reduced to a tiny fraction of gross weight, squeezed between fuel and empty weight. Thus, a relatively small error in estimated empty weight may eliminate the payload". The need for a re-evaluation of the "knock-down factors" and other similar conservative design criteria and methods is hence evident.

The large discrepancies between theoretical predictions and experimental buckling loads for thin shells, and the corresponding scatter of experimental results are attributed to three causes:

1. Initial Imperfections.
2. Boundary Effects.
3. Inelastic Effects.

Cylindrical shells under axial compression have been the focus of most studies since they have probably exhibited the largest discrepancies and scatter (see for example Refs. 3 - 5).

In the last decade initial geometrical imperfections have been accepted as the main degrading factor on the buckling loads of thin shells. In 1963 Koiter applied the initial postbuckling analysis, first proposed by him in 1945<sup>6</sup>, to a cylindrical shell with a special axisymmetric initial imperfection<sup>7</sup> and showed that even very small imperfection amplitudes resulted in very significant reductions of the buckling load. The many investigations that followed have recently been reviewed in a survey of postbuckling theory by Hutchinson and Koiter<sup>8</sup>, which shows the importance attributed to geometrical imperfections by all the investigators. This emphasis, however, appears to have overshadowed the other important effects. Even in the case of initial imperfections, usually only geometrical imperfections have been considered and practically no attention has been given to initial material imperfections, though they too may reduce the buckling loads appreciably (see for example Ref. 9 where residual stresses were shown to be a prime cause of reduction in the buckling pressures of welded stainless steel conical shells).

After devoting much effort to studies of the influence of in-plane boundary conditions, as well as prebuckling deformations, on the buckling of cylindrical shells under axial compression and other loads in the beginning of the sixties (see for example Refs. 3, 10-19), most investigators then relegated the boundary conditions to second place. Only recently has there been renewed interest in their effects (see for example Refs. 20, 21, 22).

The usual buckling analyses of thin shells assume linear elastic stress-strain laws and imply therefore that instability occurs at a critical stress appreciably below the yield stress. Plastic buckling is related only to relatively thick shells, for example cylindrical shells with  $(R/h) < 100$ . In 1969, however, Mayers and Wesenberg<sup>23</sup> called attention to the importance of non-linear material behavior in the failure mechanism of imperfect axially compressed cylindrical shells, even when their average compressive stress remains below the limit of proportionality. Inelastic effects may combine with initial imperfections to yield considerable reductions in load carrying capability. They represent, therefore, a prime cause of experimental scatter. The imperfection sensitivity aspects of plastic buckling of shells has also been emphasized in a recent survey by Hutchinson<sup>24</sup>.

Now for stiffened shells, and in particular closely stiffened shells (for which local panel buckling is rarely critical), the effect of geometric imperfections is less pronounced. Hence the reduction in predicted buckling loads and scatter of test results is less severe, provided the boundary conditions are adequately accounted for<sup>1, 25, 26</sup>. The influence of the boundary conditions is, however, found to be of prime importance in stiffened shells. The inelastic effects are also more significant in stiffened shells than in unstiffened ones, on account of their smaller effective radius to thickness ratios<sup>27</sup>. Hence in stiffened shells the three factors appear to be of the same importance, with boundary effects sometimes even dominant. The influence of boundary conditions has been the subject of a recent study by the authors<sup>28</sup> and will be extended here.

The importance of stiffened shells in design of



aerospace vehicles has resulted in continuous efforts to revise and improve design criteria. The recent NASA design manuals<sup>29,30</sup> have been supplemented by many reassessments and new criteria (see 26, 31 - 33). Design criteria and methods for buckling and vibrations of stiffened shells will also be proposed in the paper and their influence on structural efficiency will be discussed.

The vibrations of thin shells have also been investigated extensively, as is evident from a recent NASA survey<sup>34</sup>. The vibrations of shells subjected to static loads have, however, received much less attention. Some recent results on the vibrations of axially loaded stiffened cylindrical shells<sup>35-37</sup> will therefore be reviewed. Though the influence of boundary conditions has usually been taken into account in shell vibrations, that of in-plane boundary conditions for stiffened shells has only lately been considered by the authors<sup>28,38</sup> and will be briefly discussed. The effect of initial imperfections on vibrations has only received little attention (see for example Ref. 39) and will be discussed. Inelastic effects will not be considered for vibrations.

Except for some brief references to parallel results in stiffened conical shell, the discussion is limited to ring and stringer-stiffened cylindrical shells, with emphasis on axial compression loading and vibrations of axially compressed shells.

## II. Linear Theories and Correlation with Buckling Tests.

If classical linear theory can be assumed to predict buckling loads in shells with closely spaced stiffeners adequately, as already stated by van der Neut in 1962<sup>40</sup>, the bounds of validity of this assumption and the interpretation of "adequate" emerge from the experimental results discussed later. The general instability of stiffened cylindrical shells can be analyzed with a simple "smeared" stiffener theory derived in 1963<sup>41</sup> which takes into account the eccentricity of stiffeners. Because of its simplicity, this theory has been used by many investigators (see for example Ref. 42) and has been applied to conical and spherical shells (for example Refs. 43 or 44). Its results also correlate well with those obtained by the earlier theory of van der Neut<sup>45</sup>. The analysis is presented in detail in Ref. 46 for the case of axial compression, and is outlined in Refs. 1 and 47 where the detailed analyses for other loading cases are also referenced.

The analysis employs the linear stability equations (usually Donnell Equations), into which the "smeared" rings and stringers are introduced through the force and moment expressions when the equations are represented in terms of displacements. In the mathematical model the stiffeners are "smeared" to form a cut layer. For example, external rings are replaced by a layer of many parallel rings that cover the whole outside of the shell, touch each other but are not connected to each other. The main assumptions are therefore, as in Refs. 41 or 46

- The stiffeners are "distributed" over the whole surface of the shell.
- The normal strains  $\epsilon_x(z)$  and  $\epsilon_\phi(z)$  vary linearly in the stiffener as well as in the sheet. The normal strain in the stiffener and in the sheet are equal at their point of contact.
- The shear membrane force  $N_{x\phi}$  is carried entirely by the sheet.
- The torsional rigidity of the stiffener cross-section is added to that of the sheet (the actual increase in torsional rigidity is larger than that assumed).

The middle surface of the shell is chosen as reference line and the expression for forces and moments in terms of displacements are:

$$\begin{aligned} N_x &= [Eh/(1-\nu^2)] [u_{,x}(1+\mu_1) + \nu(v_{,\phi} - w) - \chi_1 w_{,xx}] \\ N_\phi &= [Eh/(1-\nu^2)] [(v_{,\phi} - w)(1+\mu_2) + \nu u_{,x} - \chi_2 w_{,\phi\phi}] \\ N_{x\phi} = N_{\phi x} &= [Eh/2(1+\nu)] (u_{,\phi} + v_{,x}) \\ M_x &= -(D/R) [w_{,xx}(1+\eta_{01}) + \nu w_{,\phi\phi} - 12(1-\nu^2)(R/h)^2 \chi_1 u_{,x}] \\ M_\phi &= -(D/R) [w_{,\phi\phi}(1+\eta_{02}) + \nu w_{,xx} - 12(1-\nu^2)(R/h)^2 \chi_2 (v_{,\phi} - w)] \\ M_{x\phi} &= +(D/R) [(1-\nu) + \eta_{t1}] w_{,x\phi} \\ M_{\phi x} &= -(D/R) [(1-\nu) + \eta_{t2}] w_{,x\phi} \end{aligned} \quad (1)$$

where  $\mu_1, \mu_2, \eta_{01}, \eta_{02}, \eta_{t1}$  and  $\eta_{t2}$  are the changes in stiffnesses due to stringers and frames and  $\chi_1$  and  $\chi_2$  are the changes in stiffnesses caused by the eccentricities of the stringers and rings.  $u, v$  and  $w$  are the additional displacements during buckling and they are non-dimensional, the physical displacements having been divided by the radius of the shell.

Substitution of Eqs. (1) into the Donnell stability Equations yields them in a simple form in terms of displacements (see Eqs. (2) of Ref. 1, Eqs. (12) of Ref. 41, or Eqs. (13) of Ref. 46).

For classical simple supports (SS3) the same displacements as for isotropic cylinders also solve these equations for stiffened cylinders. The same is true also for classical clamped ends (C2), though the similarity applies here to displacements and solution by the Galerkin method, since a closed form of solution is not possible. For other in-plane boundary conditions, improved analyses have also been developed (see for example Ref. 48 and 49). Other analyses start with slightly different assumptions than Eqs. (1) (see for example Refs. 50 or 51), but the latter are now commonly accepted for smeared stiffeners. The model (represented in the case of the Donnell theory by Eqs. (1)) can obviously be applied also to other linear and non-linear theories, with appropriate modifications (for example, a Flügge-type theory<sup>36</sup>) and appears in many computer codes (for example, in finite difference program BOSOR 3<sup>52</sup>).

The main results of linear smeared stiffener theory, relating to the effectiveness of stiffeners and relative importance of shell and stiffener parameters, are outlined in Ref. 1. One may re-

capitulate that the shell geometry may be represented by one parameter, the Batdorf parameter  $Z = (1-\nu^2)^{1/2} (L^2/Rh)$ , and the stiffener parameters are spacing, shape, cross-sectional area and eccentricity (where spacing is determined by local buckling and is therefore outside the realm of the smeared stiffener theory).

The cross-sectional area of the stiffeners or the corresponding non-dimensional area ratio (e.g.  $A_1/b_1h$  for stringers), is usually the prime geometric parameter determining their effectiveness, and has also been shown experimentally to be the prime stiffener parameter which determines the applicability of linear theory. It is however also the one that affects the weight directly, and hence structural efficiency considerations may sometimes dictate relatively weak stiffeners. The shape of the stiffeners is very important since it determines their bending and torsional stiffeners and their eccentricity. Whereas the influence of the bending stiffness is self-evident, the importance of the torsional stiffness should be stressed, in particular in the case of axial compression loading or torsion (see for example Refs. 46, 53 or 54). The stiffener eccentricity effect - the influence of positioning of stiffeners on the outside or inside of the shell - has been widely discussed and is well known. It is most pronounced for stringers<sup>46,47,55</sup>, but also important for rings<sup>46,47,56</sup>. One should remember that the eccentricity effect depends very strongly on the shell geometry and that an inversion of the eccentricity effect occurs at certain values of  $Z$ , depending on the loading and on stiffener geometry<sup>46,56</sup>. It may be noted that in general there is strong interdependence between the effects of stiffener and shell geometry and a weak one between the different stiffener geometric parameters.

The boundary effects in smeared stiffener theory will be discussed later in connection with more recent results.

Since smeared stiffener theory is valid only if the discreteness of stiffeners is negligible, this effect has been studied by many investigators (see bibliography in Ref. 57 or 58). A convenient discrete stiffener theory is obtained by consideration of stiffeners as linear discontinuities represented by the Dirac delta function instead of being "smeared". The force and moment expressions of Eqs. (1) are modified accordingly and the remainder of the analysis is similar to the smeared stiffener theory.

For cylindrical shells with discrete rings buckling under hydrostatic pressure appreciable load reductions are found even when the number of rings is not small<sup>57</sup>. In the case of axial compression, the discreteness effect of rings is usually negligible (see Refs. 25 and 59). If the number of rings in the shell is very close to the number of axial half-waves in the predicted axisymmetric buckle pattern, a recheck should be made with discrete stiffener theory. Such a recheck in typical shells<sup>59</sup> confirmed, however, the negligibility of discreteness effects except when local buckling was dominant.

In the case of stringer-stiffened shells the discreteness effect is again negligible for practical stringer spacings required to ensure failure by general instability<sup>58</sup>. Significant discreteness

effects are only likely for stringers with very large torsional stiffness, in shells with  $Z < 400$ .

The survey of the experimental evidence in 1972 confirmed earlier statements about the adequacy of linear theories<sup>40,25</sup> and led again to the conclusion<sup>1</sup> that: "Classical linear theory is applicable to integrally ring-and-stringer-stiffened cylindrical shells as a first approximation, with the same reliability as for isotropic shells under external pressure". Since the completion of that survey, many additional shells were tested in the framework of the study on the vibrations of stiffened shells and their correlation with instability. Hence the collected experimental evidence can be updated and reviewed. Unfortunately only tests carried out at the Technion Aircraft Structures Laboratories are included in the updating, since insufficient data is available at present on recent test results obtained by other investigators<sup>60,61</sup>. The shells tested in Ref. 61 have apparently also too low an  $(R/h)$  to be representative of aerospace structures. Further updating of the collected experimental results will therefore be necessary.

The applicability of linear theory is conveniently expressed by the ratio of the experimental buckling load  $P_{exp}$  to that predicted by linear theory  $P_{cr}$ , called by the senior author<sup>25</sup> "linearity"  $\rho = (P_{exp}/P_{cr})$ , in preference to the term "knock down" factor used in unstiffened shells, since in closely stiffened shells  $\rho$  is usually closer to unity. In view of the importance of the boundary conditions, especially in stringer-stiffened shells, the results will be reclassified according to B.C.'s and the scatter will be reduced, in another section of the paper, by application of the correlation with vibration tests, proposed recently by the authors<sup>28</sup>.

For ring-stiffened shells under axial compression the boundary effects are, however, relatively small, and hence the recent results for 7075-T6 aluminum alloy RO shells<sup>62</sup> are superimposed in Fig. 2 on the earlier results reproduced from Ref. 63, where  $\rho = (P_{exp}/P_{SS3})$ . Since the area ratio  $(A_2/b_2h)$  is the prime stiffener parameter,  $\rho$  is plotted as a function of it in Fig. 2 for shells of different materials and investigations<sup>25,59,62,63,64</sup>. The recent results fit within the scatter band of the earlier ones and reconfirm therefore the earlier conclusions<sup>1</sup>: For  $(A_2/b_2h) > 0.3$  there is  $\pm 10 - 15\%$  scatter about a mean  $\rho = 0.95$  (except one point that is - 17% below 0.95), or one can state roughly that  $\rho = 0.8$ . As the ring area ratio decreases below 0.3,  $\rho$  decreases, at first slowly, but below 0.15 rather rapidly. Hence applicability of linear theory appears to be bounded here by  $(A_2/b_2h) \geq 0.2$ .

The results for stringer-stiffened cylindrical shells under axial compression are now plotted in two Figures, one for simple supports, Fig. 3, and one for clamped ends, Fig. 4. The recent RO shells<sup>62</sup> are plotted together with the earlier test results from Refs. 53, 54, 55, 59, 65 and 66 surveyed in Ref. 1. Some of the earlier results are again excluded for reasons detailed in Refs. 1 or 54. The scatter in Figs. 3 and 4 is larger than in Fig. 2 and the trend less clearly defined, but the stiffener area ratio appears to be a major factor also in stringer-stiffened shells.

The "linearity" of all the shells considered as simply supported (Fig. 3), except those from

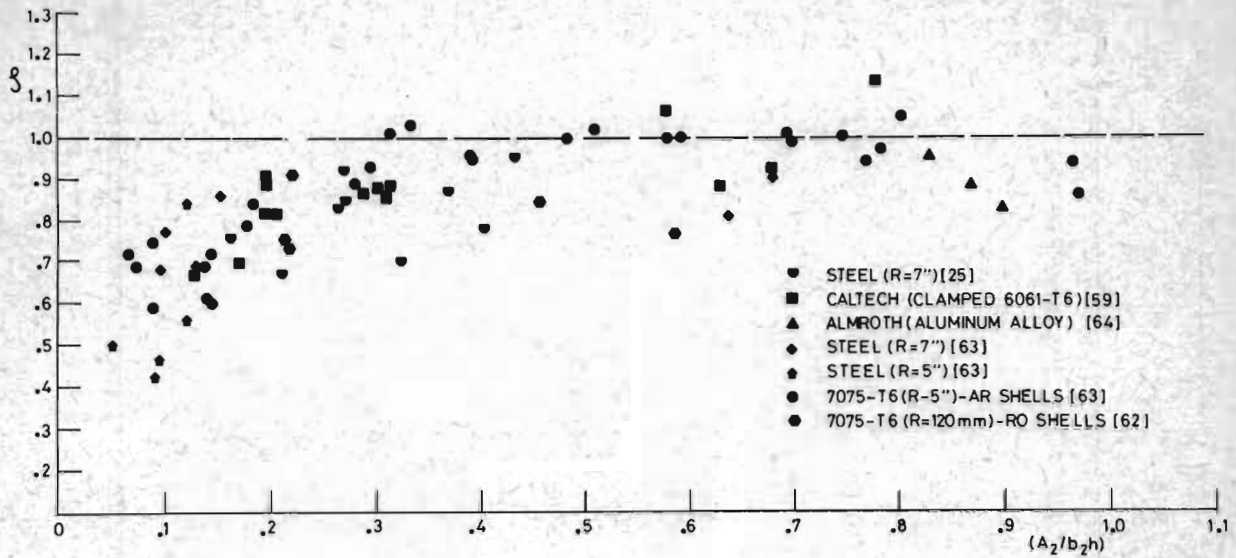


Figure 2. "Linearity" of Ring-Stiffened Cylindrical Shells under Axial Compression as Function of Ring Area Parameter.

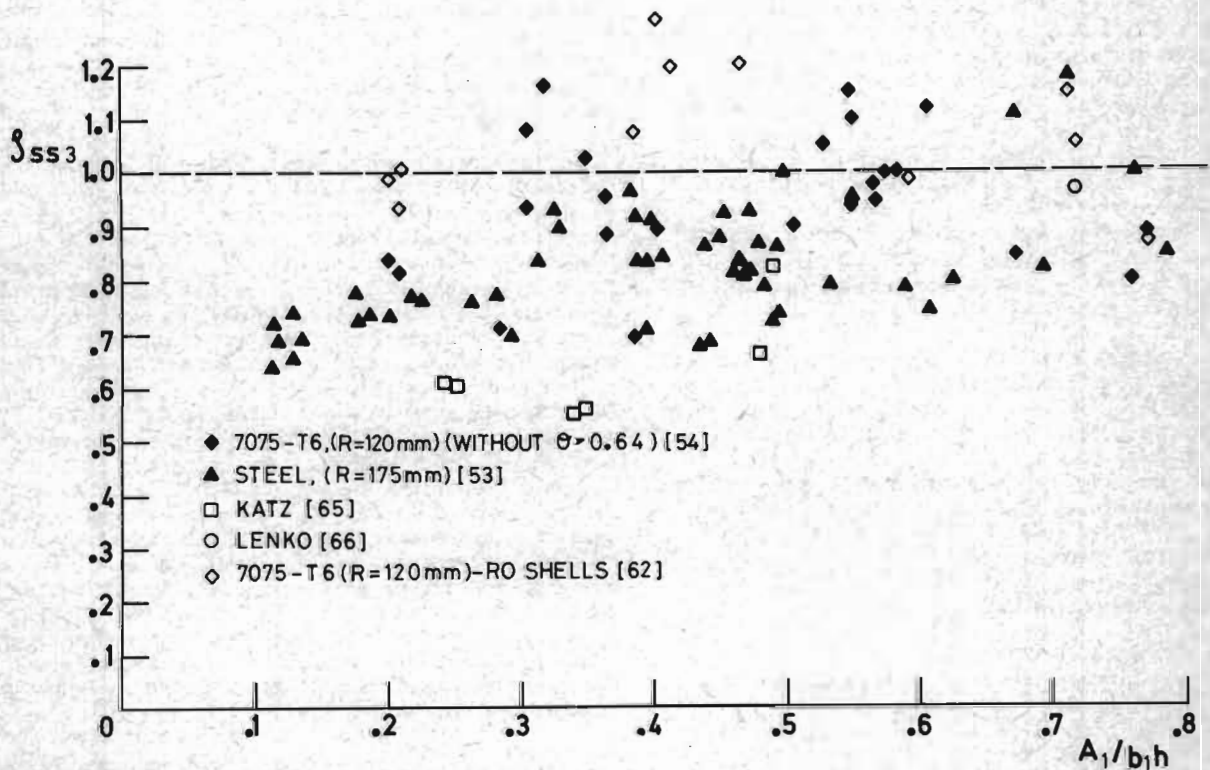


Figure 3. "Linearity" of Simply Supported Stringer-Stiffened Cylindrical Shells under Axial Compression as Function of Stringer Area Parameters.



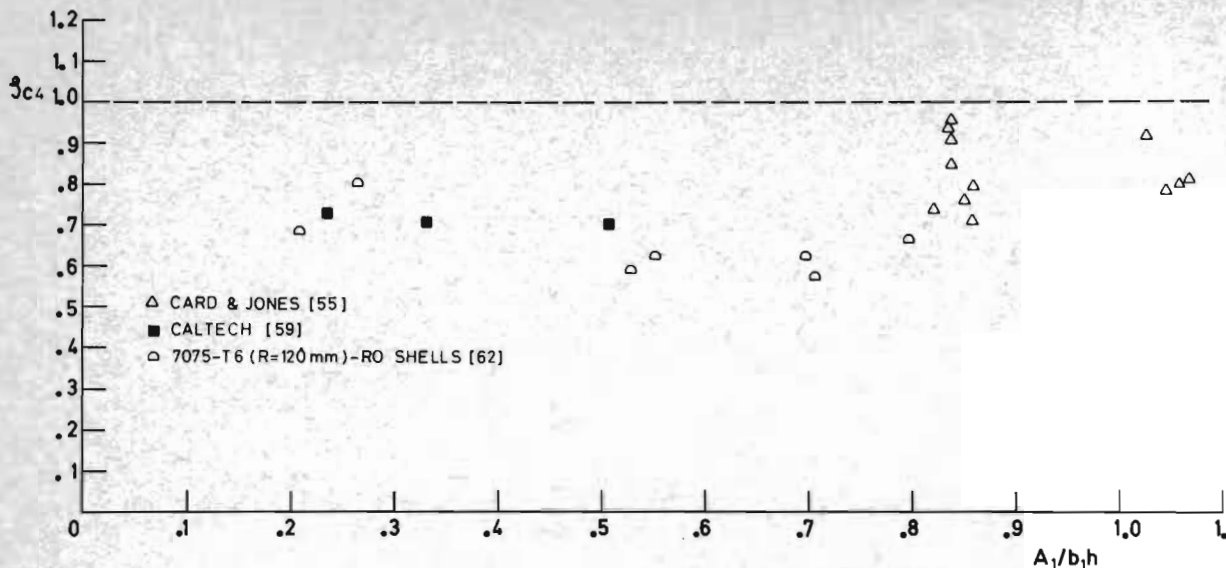


Figure 4. "Linearity" of Clamped Stringer-Stiffened Cylindrical Shells under Axial Compression as Function of Stringer Area Parameter.

Ref. 65, is above 0.65 even for weak stringers, and for  $(A_1/b_1h) > 0.5$  all shells have  $\rho > 0.72$ . In some of the shells tested by Katz<sup>65</sup>, which are the exceptions here, early local buckling was clearly observed and local buckling may have occurred also in some of the others of this series. The scatter in Fig. 3 is usually about  $\pm 20\%$ , and even  $\pm 30\%$  in regions where many tests accumulate, about an average  $\rho$  that rises from  $\rho \approx 0.7$  for weak stringers to  $\rho \approx 0.9$  for heavier ones. This scatter is partly due to incomplete definition of the boundary conditions and can be significantly reduced by a better determination of the actual boundary conditions, as will be shown in Section 8 (Fig. 17). For clamped shells (Fig. 4) lower values of "linearity", down to  $\rho \approx 0.6$ , are observed. The  $\rho$  values for some of the recent RO shells<sup>62</sup> with fairly heavy stiffening,  $(A_1/b_1h) \approx 0.5-0.7$ , are conspicuous. Though again more realistic boundary conditions correct the deviations and improve the "linearity" see (Fig. 17), the values are relatively low, which may be partly due to additional imperfections introduced by the clamping.

The influence of shell geometry and stiffener spacing has been studied in the earlier studies<sup>1,53,54</sup>. Different combinations of  $(A_1/b_1h)$  and  $Z$ , as well as of other geometric parameters have also been tried<sup>1,53,54</sup> to reduce scatter and to define trends. One more successful attempt to reduce scatter significantly has been a "weighted linearity"<sup>54</sup>  $\rho/[1 + (A_1/b_1h)]$  (see for example Fig. 10 of Ref. 1). Correlations with SS4 boundary conditions have also been tried<sup>54</sup>, and will be discussed again in Sections 8 and 10.

Experimental results for other loading cases provide additional support for the adequacy of linear theory in the analysis of the buckling of stiffened cylindrical shells. Some bending and hydrostatic pressure tests, in all of which  $\rho > 0.9$ , have also been discussed in Ref. 1 (see

for example Table 2 of Ref. 1). Tests on ring-stiffened conical shells<sup>67</sup> also support the conclusions arrived at for cylindrical shells (see for example Fig. 12 of Ref. 25).

### III. Vibrations of Stiffened Cylindrical Shells.

In addition to their well-known importance for determination of the dynamic response of shells, free vibrations have proven to be a valuable tool for determination of boundary conditions and correlation with buckling of shells. This also emerged from the theoretical and experimental studies on the vibrations of stiffened cylindrical shells carried out by the authors<sup>28,35-37</sup>.

The theoretical studies employ a "smeared" stiffener theory developed for buckling analysis<sup>41</sup> which considers the eccentricity of the stiffeners. For vibrations this theory was first used in unloaded shells by McElman et al.<sup>68</sup>. It was later compared by Parthan and Jones<sup>69</sup> with a discrete stiffener analysis and found to be adequate for uniformly and closely spaced stiffeners. They also showed that while in-plane inertia cannot be neglected, rotational inertia is negligible. Hence the analysis developed considers only radial and in-plane inertias. The details are given in Ref. 36 and in an abbreviated form in Ref. 35. Three sets of Equations and boundary conditions were derived. The first set follows Donnell theory, the second set treats the small vibration displacements according to the more exact Flugge theory, but takes into account the contribution of the static stresses according to Donnell theory, the third set follows the Flugge theory throughout. The three sets can be written in a concise form which brings out the differences between them. For example the equilibrium equation in the axial direction is:

$$\begin{aligned} & \{1 + \mu_1 + \frac{[P(1-\nu^2)/2\pi ERh]\underline{u}}{,xx} + \frac{[(1-\nu)/2](1+\underline{T})u}{, \phi\phi} + \\ & + \frac{[(1+\nu)/2]v}{,x\phi} - \underline{vw}_{,x} - (X_1 - \underline{T})w_{,xxx} - \frac{T[(1-\nu)/2]w}{,x\phi\phi} = \\ & = [R^2 M(1-\nu^2)/Eh]u_{,tt} \end{aligned} \quad (2)$$

and similarly for the two other equations and the boundary conditions, where the terms in Eq. (2) underlined twice are those appearing in the third set only, those underlined once appearing in the second and third sets but not in the first one. The systems of equilibrium equations and boundary conditions are solved by the method usually referred to as the "exact method", used extensively for isotropic shells (for example in Ref. 70). Many calculations were performed with the three sets<sup>36</sup> and for the range of geometries studied only very small differences were found between the results of the three analyses. The results discussed in this Section were obtained with the Donnell-type theory for stringer-stiffened shells and with the second set of equations for ring-stiffened ones. The results of later tests discussed in other Sections were all obtained with the Flugge-type theory.

Since buckling may also be defined as the state of vanishing frequency of free vibrations, the analysis and the resulting computer program also yield buckling loads. The results of analysis for vibration and buckling were found to agree well with those obtained by other methods<sup>36</sup>. The present method has the advantage that it can be readily applied to different combinations of boundary conditions including elastic restraints<sup>38</sup>.

Concurrently with the theoretical study vibration tests were carried out. The test apparatus and procedure is described in detail in Ref. 37 and summarized in Ref. 35. The shell is excited by an acoustic driver which is inside the shell. The response of the shell is measured by a microphone outside the shell. The excitation frequency is changed and resonance is detected by the help of Lissajous figures. When a resonance frequency is detected, the mode of vibration is recorded by plotting the microphone reading versus its circumferential or axial position on X-Y recorders. The load is applied with a screw-jack and the load distribution is checked by an array of pairs of strain gages, which permit separation between compressive and bending strains. The first test series<sup>37</sup> included 4 stringer-stiffened shells, (RO-2 to RO-5), 2 ring-stiffened ones (RO-6, RO-7) and one isotropic shell (RO-1). The dimensions of the shells are given in Table 1 of Ref. 35 or Table 2 of Ref. 37. Their (R/h) = 460-500, (L/R) ≈ 2, area ratio (A<sub>1</sub>/b<sub>1</sub>h), or (A<sub>2</sub>/b<sub>2</sub>h) ≈ 0.2 - 0.5 and eccentricity of stiffener (e<sub>1</sub>/h), or (e<sub>2</sub>/h) ≈ -1.5 to -3.0. The specimens were cut from 7075-T6 aluminum alloy drawn tubes, with E = 7500 kg/mm<sup>2</sup> (10.6 × 10<sup>6</sup> psi), σ<sub>0.1%</sub> = 54 kg/mm<sup>2</sup> (76700 psi) and ν = 0.3. The shells with their integral stiffeners were accurately machined by a process described in Ref. 54. These tests were carried out with the specimens clamped in a circular groove of the same inner diameter and the outside gap filled with low melting point Wood-metal or Cerrobond<sup>37</sup>.

Curves of frequency versus applied load were plotted with the experimental results and compared with theoretical predictions for C4 B.C.'s. As an example, the curves for two stringer-stiffened shells

RO-2 and RO-4 (which are twins of practically identical dimensions) are reproduced from Ref. 35 and presented in Fig. 5. The curves presented describe

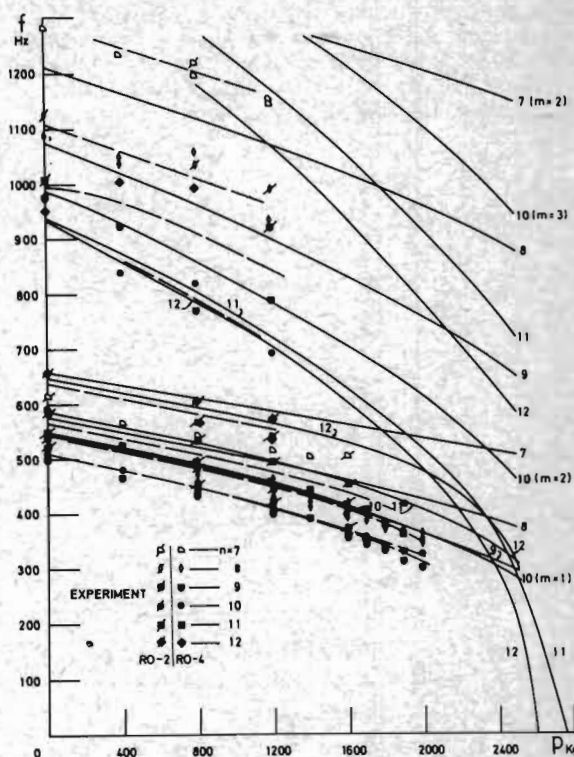


Figure 5. Vibrations of Loaded Stringer-Stiffened Cylindrical Shells - Frequency Versus Axial Load of Shells RO-2 ((R/h)=48; (L/R)=1.98; (A<sub>1</sub>/b<sub>1</sub>h)=0.225; (I<sub>11</sub>/b<sub>1</sub>h<sup>3</sup>)=0.082; (e<sub>1</sub>/h)=-1.54; η<sub>11</sub>=0.91) and RO-4 ((R/h)=498; (L/R)=1.97; (A<sub>1</sub>/b<sub>1</sub>h)=0.265; (I<sub>11</sub>/b<sub>1</sub>h<sup>3</sup>)=0.125; (e<sub>1</sub>/h)=-1.69; η<sub>11</sub>=1.35) for n = 7-12 (reproduced from Ref. 35)

vibration modes with circumferential waves n = 7-12 and axial half-waves m = 1, 2 and 3. Curves were also fitted to the experimental points, wherever sufficient results were available. One may note in Fig. 5 (and this can also be observed in the many curves plotted in Ref. 37) that the experimental results show better defined trends for m = 1 than for m = 2. In Fig. 6 (again reproduced from Ref. 35) an example for a ring-stiffened shell RO-7 is given for the modes n = 6-10 and m = 1, 2. For the ring-stiffened shells the experimental results exhibit well defined trends also for m=2<sup>35</sup>.

This first test series<sup>37</sup> confirmed the adequacy of the smeared stiffener analysis<sup>36</sup>, for the range of geometries studied. Good agreement was obtained between experiment and theory. In all the tests (except for one mode in one test) the measured frequencies and the experimental buckling loads were lower than the calculated ones. The study of frequencies and modes shapes showed that whereas at zero load for a given circumferential wave number the low frequency was connected with one axial half-wave, the second with two half-waves etc; with

increasing axial load the low frequencies appeared for two or more axial half-waves which correspond also to the buckling mode of the shell. The behavior of stringer-and ring-stiffened shells differs. In stringer-reinforced shells, due to the increased axial bending stiffness, the modes of low frequencies and the buckling modes occur with relatively high circumferential wave numbers (see Fig. 5). In ring-stiffened shells, because of the increased circumferential bending stiffness, the low frequencies and buckling modes occur with smaller circumferential wave numbers (see Fig. 6). Near the buckling load, modes with many axial waves are predicted and buckling may also initiate in such a mode.

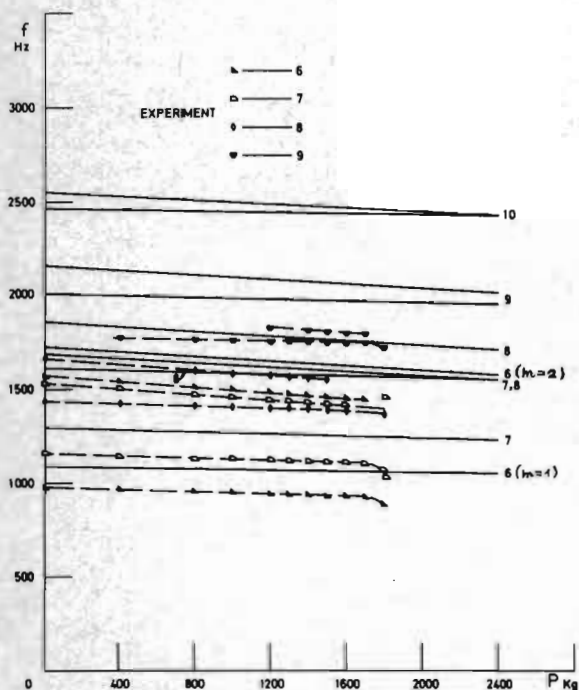


Figure 6. Vibrations of a Loaded Ring-Stiffened Cylindrical Shell RO-7((R/h)=466; (L/R)=1.93;  $(A_2/b_2h)=0.585$ ;  $(I_{22}/b_2h^3)=1.12$ ;  $-(e_2/h)=2.89$ ;  $n_{t2}=7.01$ ) for  $n=6-10$  (reproduced from Ref.35)

Many additional tests on the vibrations of axially loaded stiffened-shells have been carried out as part of the studies aimed at the definition of the boundary conditions and the prediction of buckling load discussed below (see also Refs. 38 and 71), and have confirmed these observations made on the first test series.

#### IV. Influence of Boundary Conditions

The influence of in-plane boundary conditions and prebuckling deformations on the buckling and vibrations of stiffened cylindrical shells under axial compression have been discussed in detail in <sup>28</sup> The main results obtained there will be summarized and extended by additional parametric studies. The discussion is again limited to axial compression loading. Another loading case, hydrostatic pressure,

has been discussed in Ref. 1.

For ring-stiffened cylindrical shells under axial compression the effects are similar to those in isotropic shells <sup>48,1</sup>, with heavy internal rings exhibiting the only significant deviation in that the weakening effect of the "weak in shear" boundaries tends to disappear (Fig. 5 of Ref. 48), without changing the negligibility of the effect of axial restraints.

For stringer-stiffened cylindrical shells under axial compression the influence of in-plane boundary conditions differs appreciably from that in isotropic shells - the main difference being that the axial restraint ( $u=0$  instead of  $N_x=0$ ), which has no effect in unstiffened shells of ring-stiffened ones under axial compression <sup>11</sup>, becomes the predominant factor, whereas the circumferential restraint ( $v=0$  instead of  $N_\phi=0$ ) has only a minor influence. Hence, for heavy or medium stringers, SS3 and SS1 boundary conditions yield practically identical results and similarly SS1 and SS2 hardly differ (see for example Fig. 6 of Ref. 1). The studies of Weller <sup>49</sup> show that the effects also strongly depend on the shell geometry parameter Z and on the stringer parameters  $(A_1/b_1h)$  and  $(e_1/h)$ . The effect of axial restraint is more pronounced for internal stringers, but even for external ones axial restraint ( $u=0$ , SS2 or SS4) may raise the predicted buckling load for medially or heavily stiffened shells by 50% or more if the shell is long. The parametric studies carried out <sup>28</sup> for simply supported stringer-stiffened shells, with relatively efficient stringers (their properties appear in Table 2 of Ref. 28), confirm the trends of Ref.

<sup>49</sup> and indicate that the largest effect occurs for medium stiffening, falling off as the stiffening increases further. This behavior, that for a given Z the increase in  $(A_1/b_1h)$  yields an apparent decrease in  $P_{SS4}/P_{SS3}$ , can be explained from plots versus Z, which show the shift of the axial constraint effect to a higher Z for heavier stringers <sup>49</sup>. For stringer-stiffened shells with clamped ends the effect of axial restraints are much smaller <sup>28</sup>.

Additional parametric studies have been carried out for slightly less efficient stringers, closer to the geometries of the test shell. The main geometric parameters are given in Table 1. The calculations have again been carried out with a linear theory <sup>36</sup> and with BOSOR <sup>35,2</sup> taking nonlinear prebuckling into account. The results, some of which are presented in Fig. 7, reconfirm the trends found in Refs. 28 and 49. The effect of rotational restraints is also presented in the figure. Comparison between the C4 and SS4 curves in Fig. 7 shows that, whereas rotational restraint is effective only for short shells, axial restraint is even more effective for long shells.

As noted in Ref. 28, the influence of axial constraints on the buckling of axially compressed stringer stiffened-shells resembles that observed in isotropic shells buckling under lateral hydrostatic pressure <sup>19</sup>, or vibrating freely <sup>70</sup>. The similarity with the influence of in-plane boundary conditions for vibrations is one reason for the success of the correlation between vibration and buckling tests reported in Ref. 28 and whose continuation is discussed below.

The influence of in-plane boundary conditions and rotational restraints on the vibrations of



Table 1. Stringer Geometric Parameters for Theoretical Studies.

$A_1/b_1h$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$I_{11}/b_1h^3$	0.008	0.067	0.226	0.535	1.05	1.81	2.87	4.28	6.10
$-e_1/h$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$\eta_{t1}$	0.116	0.732	1.89	3.36	5.08	6.78	8.64	10.4	12.3

$R/h = 480,$

$\nu = 0.3,$

Specific Gravity = 2.80

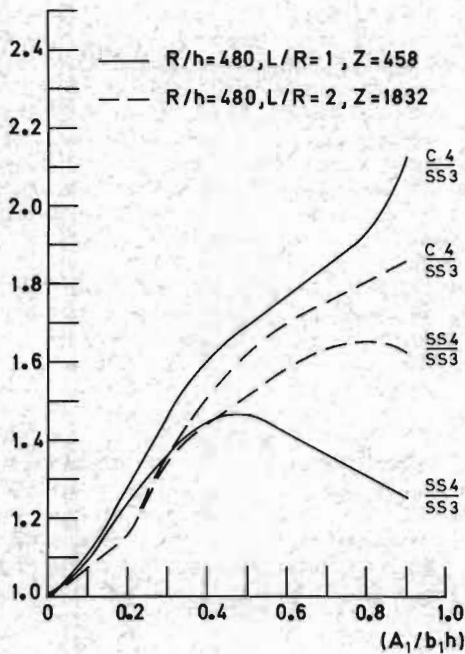


Figure 7. Effect of Axial and Rotational Restraints on Buckling of Stringer-Stiffened Shells (Stringer Geometry is Given in Table 1).

stiffened shells has been dealt with in detail in Refs. 28 and 38, and will be reconsidered below in the discussion of the continued correlation studies. The out-of-plane boundary conditions have also been discussed in Ref. 28.

The effect of prebuckling deformations, caused by the edge constraints has been extensively studied for isotropic cylindrical shells, and its dependence on shell geometry and boundary conditions for the case of axial compression has been recently summarized in the very complete parametric study by Yamaki and Kodama<sup>20</sup>. The earlier studies for stiffened cylindrical shells have been summarized in Ref. 28 where a partial parametric study has also been carried out with BOSOR 3<sup>52</sup> and the linear theory of Ref. 49. In Fig. 1 of Ref. 28 curves for the variation of the ratio of buckling load to that with prebuckling deformations neglected  $\rho_{pre} (= P_{pre}/P_{membrane})$  with shell geometry have been plotted for two types of shells with relatively efficient external stringers. The curves are for

SS3 and SS4 boundary conditions and are compared with the corresponding ones for isotropic shells (reproduced from Ref. 20). The general trends observed were: (1) a shift to higher  $Z$  with increase in stiffening; (2) a large increase in buckling loads for short shells, again more pronounced the heavier of the stringers; (3) even for long shells a smaller decrease in  $\rho_{pre}$  than in isotropic shells, and (4) a reversal of the relative magnitudes of  $\rho_{pre}$  with SS3 and SS4 B.C.'s compared to those for isotropic shells.

A parametric study has now been carried out for a typical less efficient stringer (taken from Table 1), extended also to include clamped boundary conditions. The variation of  $\rho_{pre}$  with  $Z$  is plotted for this stringer, with  $(A_1/b_1h) = 0.4, (I_{11}/b_1h^3) = 0.535, (e_1/h) = -2.5$  and  $\eta_{t1} = 3.36$ , in Fig. 8. The

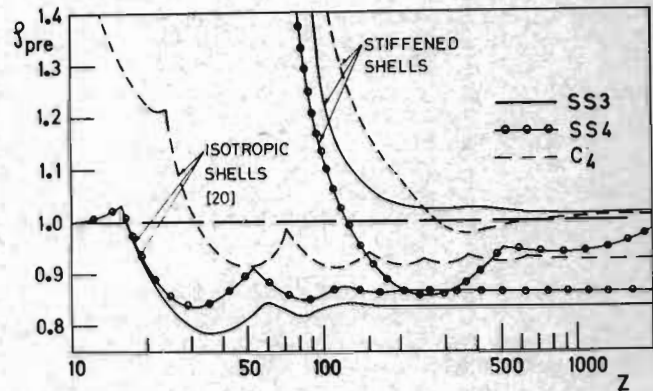


Figure 8. Effect of Prebuckling Deformation on Buckling of Stringer-Stiffened Shells - Variation with Shell Geometry ( $R/h=480; (A_1/b_1h)=0.4; I_{11}/b_1h^3=0.535; (e_1/h)=-2.5; \eta_{t1}=3.36$ ).

corresponding  $\rho_{pre}$  for isotropic shells is again plotted for comparison in the Figure, with  $C_4$  also included. Similar general trends are observed in Fig. 1 of Ref. 28, except that due to the less efficient stringers the increase in buckling loads for short shells is less pronounced and that  $\rho_{pre}$  for SS4 drops to a lower value for  $200 < Z < 500$  before stabilizing in the large  $Z$  range at a value close to 1.

The influence of nonlinear prebuckling deformation on the vibrations of axially loaded stringer-stiffened shells has been discussed in Ref.

28 . The effect has been found negligible for zero load and noticeable, though still very small for high loads. For example, for the vibration mode shown in Fig. 10 of Ref. 28 ,  $\rho_{pre} = 0.99$  for SS4 B.C's at zero load (where  $\rho_{pre}$  is the ratio of frequency obtained with nonlinear prebuckling to that obtained with linear theory), and decreases to  $\rho_{pre} = 0.93$  near buckling. The effect for SS3 is even smaller.

One may summarize the present parametric studies, those of Ref. 28 and the results of other investigators in the following practical conclusions:

- (1) Axial restraints, and also rotational restraints strongly affect the buckling loads of stringer-stiffened shells. This effect depends on shell and stiffener geometry. Similar effects are observed in the vibrations of stringer-stiffened shells.
- (2) For short shells with medium or heavy external stringers ( $Z < 250-400$  and  $(A_1/b_1h) > 0.3$  with corresponding other geometry parameters, in particular  $(e_1/h) > 3$ ), consideration of nonlinear prebuckling deformations results in substantial increase in buckling loads. Hence linear theory will yield very conservative predictions for such shells. For similar internal stringers this increase disappears for the few cases studied. Further parametric studies for internally stiffened shells are needed.
- (3) For medium and long shells ( $Z > 1000$ ) with external stringers,  $\rho_{pre} \approx 0.94-1.05$  and the effect of nonlinear prebuckling deformations may therefore be neglected.
- (4) For external stringers,  $\rho_{pre}$  for SS4 is practically always below the value for SS3 B.C's, whereas in isotropic shells  $\rho_{pre}$  for SS4 always exceeds that for SS3. This is due to the large increases in buckling loads with axial restraints (SS4) predicted by linear theory.
- (5) The influence of nonlinear prebuckling deformations on the vibrations of stringer-stiffened shells is very small and may be ignored in the cases studied.

Thus for design purposes, one can safely neglect nonlinear prebuckling deformations for externally stringer-stiffened cylindrical shells under axial compression, except in the case of very short shells, where buckling predictions would be unduly conservative.

#### V. Initial Imperfections - Their Effect On Buckling and Vibration

For unstiffened shells initial imperfections have been accepted as the main reason for the significant discrepancies between predictions and experiments. Since the initial imperfection shape in the form of an axisymmetric buckling mode assumed by Koiter in 1963<sup>7</sup> yielded very large reductions in the buckling loads, much effort has been devoted to axisymmetric imperfection theory<sup>8,31,32,61</sup>. Based on rather restrictive

imperfection distributions, it has even been argued that the effects of axisymmetric imperfection components dominate in actual cylinders and hence axisymmetric imperfection theory yields adequate design criteria<sup>32</sup>. This contention is however not upheld by results based on measurements of practical shells. The influence of asymmetric and general initial imperfections has also been studied though not as extensively<sup>8</sup>. The theoretical studies aimed at the determination of the effect of initial imperfections were mostly studies of the initial postbuckling behavior yielding the imperfection sensitivity of the structure. For unstiffened shells experimental results tend to verify the predictions of imperfection sensitivity theory qualitatively (see for example Ref. 78 ). For stiffened shells these predictions are not always verified qualitatively (see for example Fig. 31 in Ref. 54 ), since in this case other secondary effects may overshadow the imperfection sensitivity. The results of imperfection sensitivity theory can therefore not be relied upon in stiffened shells, unless all boundary effects have also been properly taken into account. For example, the large increase in imperfection sensitivity predicted by Hutchinson and Amazigo<sup>9</sup> for axially compressed stringer-stiffened cylindrical shells in the range of small  $Z$ , disappear when prebuckling deformations are taken into account in a later paper<sup>80</sup> (see Fig. 11 of Ref. 1 ).

Recently, however, emphasis has been placed on prediction of buckling loads from experimentally measured initial imperfections<sup>72,73,22</sup>, and satisfactory correlation has been achieved in some cases<sup>22</sup>. The measurement of initial imperfections for a wide range of shells is essential for a more reliable assessment of their effect on the buckling behavior. On the laboratory scale initial imperfections have been surveyed by many investigators<sup>72,74-77,59</sup>. A scan of initial imperfections for typical stringer-stiffened shell (reproduced from Ref. 59 ) is given in Fig. 9. The domination

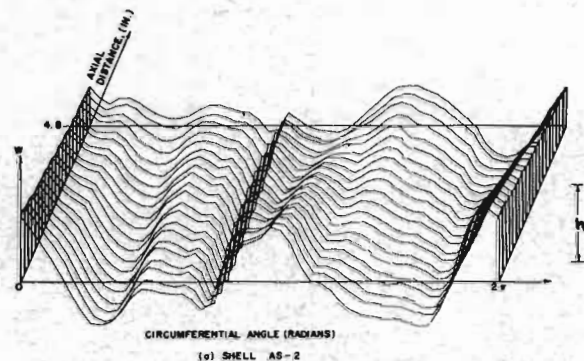


Figure 9. Typical Measured Initial Imperfections for a Clamped Stringer-Stiffened Cylindrical Shell - Shell AS-2:  $(R/h)=517$ ;  $(L/R)=1.38$ ;  $(A_1/b_1h)=0.506$ ;  $(e_1/h)=-1.72$ ; (reproduced from Ref. 59).

of large amplitude asymmetric imperfections of long axial wave length, also observed in scans of isotropic shells<sup>72</sup>, is apparent in the Figure. These



long wave length imperfections have been brought to focus only by the measurements and have not been considered in Ref. 32 and in other studies. The growth of imperfections with load has also been studied<sup>72,75,59</sup>. Additional imperfection measurement facilities are being put into operation at the Aircraft Structures Laboratory of the Technion and at other laboratories, and measurements on full scale have commenced at Georgia Institute of Technology and at the NASA Langley Research Center. An extensive imperfection measurement effort will be necessary to yield meaningful correlations between manufacturing processes and dominant imperfections, as well as between scans and the buckling loads.

The earlier investigator apparently did not attach any importance to the influence of imperfections on the vibrations of shells. Motivated by the possibility of a correlation between the effect of imperfections on vibration and buckling, the authors carried out a preliminary study<sup>39</sup> that is essentially an extension of Koiter's 1963 analysis for buckling<sup>7</sup>. The radial inertia term was added to Koiter's formulation and the effect of a similar special axisymmetric initial imperfection on the vibrations of an axially compressed isotropic cylindrical shell was studied. Though the shape of the initial imperfections and the mode of vibration are not realistic, the results of this preliminary study are of interest. It appears that initial imperfections have a significant effect on the vibrations, not only at high compressive loads as expected, but also at zero axial load. As can be seen in Fig. 10 where the

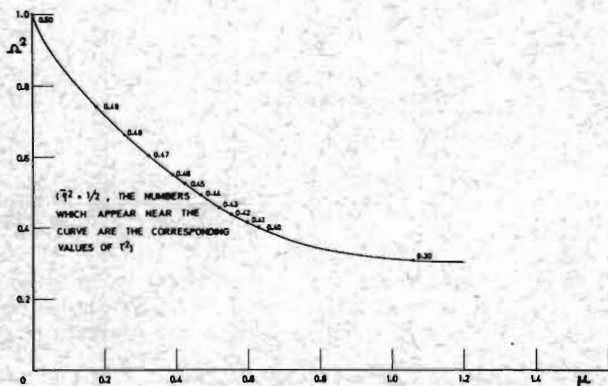


Figure 10. Vibrations of Imperfect Cylindrical Shells - Minimum Frequency Parameter versus Amplitude of Axisymmetric Imperfections at Zero Loaded (reproduced from Ref. 39)

square of the non-dimensional frequency  $\Omega^2$  is plotted versus the imperfection amplitude  $\mu$ , (reproduced from Ref. 39), an imperfection amplitude of 0.2 shell thickness, for example, reduces the square of the frequency to 73% of that of the perfect shell, and one of 0.5 shell thickness to about 48%. Though less than the corresponding reductions in buckling load predicted by Koiter<sup>7</sup> (which would be to 44% for  $\mu = 0.2$  and to 27% for  $\mu = 0.5$ ), this is a very significant reduction. After this preliminary study was accepted for publication, a similar study published recently in the USSR<sup>81</sup> came to the notice of the authors. Results of a similar nature were

obtained in that study.

The results motivated further studies by the authors of the effect of asymmetric imperfections on the vibrations of cylindrical shells, which will soon be reported. Furthermore, one is led to the tentative conclusion that the correlation of boundary effects between vibrations and buckling of stiffened shells, discussed later, includes also some correlation of the initial imperfections.

## VI. Inelastic Effects

As pointed out by Mayers and Wesenberg<sup>23,27</sup>, considerable reductions in the load carrying capability may occur in axially compressed shells even of fairly high (R/h), as a result of the combination of initial imperfections and material nonlinearity, when the material stress-strain curve becomes nonlinear at relatively low loads. These inelastic effects, or rather material nonlinearity effects, occur when the average compressive stress is still well below the usual 0.1 or 0.2 percent yield stress, and therefore plasticity correction factors would usually not be applied. Many materials like nickel, copper, some stainless steels, annealed metals or some of the advanced composites exhibit stress-strain curves with early nonlinear behavior, characterized by low exponents in the Ramberg-Osgood representation<sup>83</sup>, and shells made of them are therefore prone to these effects.

The maximum strength analysis<sup>23,27</sup> employed to study these effects, proved very successful in explaining anomalous experimental results obtained for buckling of steel stringer-stiffened shells<sup>53</sup>. There, one group of shells with smaller (R/h) gave consistently lower values of "linearity"  $\rho$  than another group of steel shells with a larger (R/h) (see Fig. 11 of Ref. 53). Since all the specimens were manufactured in a similar manner, the opposite trend had been expected, as a result of the expected increase in influence of imperfections with larger (R/h). Mayers suggested looking again at the material properties, and indeed it was found that the material of the smaller radius shells exhibited early nonlinear behavior (see Fig. 8 of Ref. 53 or Fig. 4 of Ref. 82), though its 0.1% and 0.2% yield points were well above the buckling stresses. Mayers and Meller<sup>82</sup> then applied their maximum strength analysis to two typical shells of each group. The results (see Table 3 of Ref. 1, Table 3 of Ref. 54, or Table 2 of Ref. 82) show that in the case of the two shells with smaller (R/h) the correction for inelastic effects raises the "linearity" considerably, since the material exhibits early nonlinear behavior, whereas in the two with the larger (R/h) it is negligible, on account of the relatively linear stress-strain curve. The trend was hence reversed and the anomaly removed.

Since increased structural efficiency also means higher stresses in stiffened shells, the inelastic nonlinear material effects will become more important in optimization studies. This is evident in the sample study<sup>82</sup> based on the stringer-stiffened aluminum shells tested by Card<sup>55</sup>. It may also be expected that for stiffened shells, imperfection sensitivity studies in the plastic range, like that carried out by Hutchinson<sup>24</sup> for isotropic cylindrical shells, will not uphold the validity of elastic analysis observed in the unstiffened shell.



## VII. Eccentricity of Loading and Non-Uniformity of Loading

Eccentricity of loading, usually defined as the radial distance between the line of axial load application and the shell mid-skin, has been shown to have considerable influence on the buckling load of stringer-stiffened shells (see, for example Refs. 84, 85 or 86). In Ref. 86 the theoretical investigation was amplified by tests on integrally stringer-stiffened cylindrical shells loaded eccentrically and having different boundary conditions. These studies have been extended<sup>28,87</sup> to consider the influence of load eccentricity on the vibrations of axially loaded stringer-stiffened shells both theoretically and experimentally, and correlate the results with those for buckling. The details of the calculations and experiments are given in Ref. 87 and some of the results are discussed in Ref. 28. The salient features will be briefly reviewed here.

In the tests the eccentricity of loading is achieved by applying the load through the stringers. Specimens are therefore manufactured with three kinds of edges, as shown in Fig. 11 (reproduced from Ref. 28). In the case of edge A, the load is applied through the mid-skin of the shell, for edge B the load is applied through an intermediate point along the depth of the stringers,

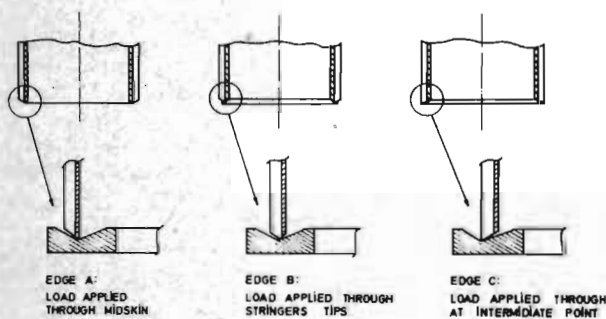


Fig. 11. Details of Load Application for Simply Supported Stringer-Stiffened Cylindrical Shells with Different Types of Edges. (Reproduced from Ref. 28)

and in the case of edge C through the tip of the stringers. In all the cases special support rings (see Fig. 11) which restrain the radial displacement of the shell edge or stringers, are accurately fitted to the shell edges.

The vibration frequencies for various load eccentricities and their variation with axial load were calculated for SS3 and SS4 B.C.'s and compared with experimental results. For SS3 B.C.'s, no influence of load eccentricity is predicted at zero axial load, and a small effect appears as the load increases, yielding higher frequencies with increasing outward load eccentricity. For  $n > 9$  the trend reverses and the frequency decreases for large outward load eccentricity. For SS4 B.C.'s, on the other hand, large differences in frequencies with load eccentricity appear already at zero load, whereas the change with axial load is less pronounced than in the SS3 case. The importance

of taking the real point of load application into account should be noted<sup>87</sup>. The experimental results<sup>87</sup> show mostly fairly clear trends and indicate that the boundary conditions are between SS3 and SS4.

The influence of eccentricity of loading on the buckling of a family of three moderately stiffened shells is shown in Fig. 12 (reproduced from Ref. 28). The detailed geometries are given there<sup>28</sup> and they fall within the range of geometries outlined in Section 8. Generally similar results were obtained for a family of heavily stiffened shells (see Fig. 11 of Ref. 28). The theoretical curves for SS3 and SS4 have a maximum at a relatively low value of load eccentricity and then decrease with increasing outward eccentricity. This behaviour correlates

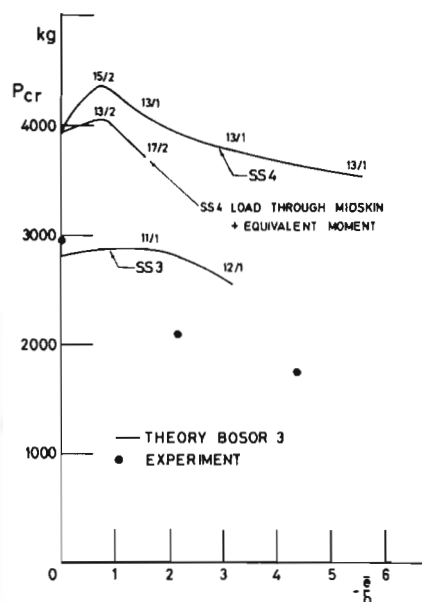


Figure 12. Influence of Eccentricity of Loading on Buckling Loads, "Medium" Stringers ( $A_1/b_1h = 0.38$ ) (reproduced from Ref. 28).

with a similar one for the vibrations<sup>87</sup>. The experimental results show fair agreement with theory. They and those of the heavy stringers reconfirm and emphasize the behavior observed in Ref. 86 that increase in outward load eccentricity results in lower buckling loads but also in a much less violent buckling phenomenon. It should be noted that unlike the initial geometrical imperfections, the residual stresses or the actual boundary conditions, the eccentricity of loading is readily determined and controlled by the designer. He must, however, remember to account for the effect of the load eccentricity and that this effect depends also on the boundary conditions. A reliable definition of the boundary conditions has therefore additional value.

Eccentricity of loading is one kind of imperfection related to loading. Another is non-uniformity of the distribution of loading along the shell edges. The effect of non-uniform axial compression on the buckling of isotropic cylindrical

shells has been the subject of many theoretical studies. For example, a summary is given in a recent study<sup>88</sup>, which shows that, not only for SS3 B.C.'s but also for SS1 B.C.'s, the critical axial compressive stress for a non-uniform load distribution is practically the same as that for a uniform one, provided certain restrictions are adhered to.

Experimentally, uniformity of loading is quite difficult to accomplish, and few reliable load distribution measurements have been carried out<sup>89</sup>. In the present tests of the RO shells the load distribution was measured. An array of five pairs of strain gages, uniformly distributed over the circumference of the shell at approximately one third of the length of the shell, was bonded to the inside and outside, in order to permit separate readings of direct and bending stresses. The measured distribution was fairly uniform for most shells, with a deviation from the mean value at high load level of less than 12%. There were however isolated cases of a deviation of up to 40%. Correlation between non-uniformity and experimental buckling loads has been attempted but the results are as yet inconclusive. Further measurements and studies are planned.

#### VIII. Experimental Definition of Boundary Conditions by Correlation with Vibration Tests

The importance of exact definition of the boundary conditions for buckling of stringer-stiffened shells has become evident in the previous Sections. The similarity of the influence of B.C.'s on buckling and vibrations of such shells, in particular for the lower natural frequencies, motivated development of a correlation technique with vibration tests to achieve this definition<sup>28</sup>. For columns, many correlation studies have been developed for assessment of the elastic restraints provided by the boundaries from vibration tests (see Ref. 90 and bibliography in Ref. 28). Though for isotropic shells no analogous methods have apparently been developed, observation of the similarity in behavior of the stringer-stiffened shells in buckling and vibration has led to the development of this technique in Refs. 28,38. The crucial fact, which makes the technique feasible in these shells, is that the buckling mode is similar to the lowest, or one the lowest, vibration modes, both in theory and experiment. The correlation with vibration tests may also be employed to reduce experimental scatter in buckling loads of stringer-stiffened shells. This was shown in detail by examples<sup>28</sup>.

This method is now briefly reviewed, for a typical shell RO-31, mentioned also in Ref. 28, the geometrical properties of which are:  $(R/h)=482$ ;  $(L/R)=1.79$ ;  $(A_1/b_1h)=0.80$ ;  $(I_{11}/b_1h^3)=4.24$ ;  $(e_1/h)=-4.48$  and  $\eta_{t1}=10.5$ . This shell is nominally clamped, and the real boundary conditions are incomplete clamping between SS4 and C4. The influence of elastic rotational restraints  $k_4$  on buckling and vibrations can be calculated with the theory of Ref. 36, extended to include elastic restraints at the boundaries<sup>33</sup>. Fig. 13 shows the variation of buckling load, between SS4 and C4 boundary conditions for this shell. The variation between SS4 and C4 is obtained with a torsional spring (the other B.C.'s are  $u=0, v=0, w=0$ ), the stiffness of which  $k_4$  is zero for SS4 and infinity for C4 B.C.'s.

Correlation with the natural vibrations of the

corresponding axially loaded shell is employed to assess the real boundary conditions of the experiment. In Fig. 14 the influence of elastic rotational

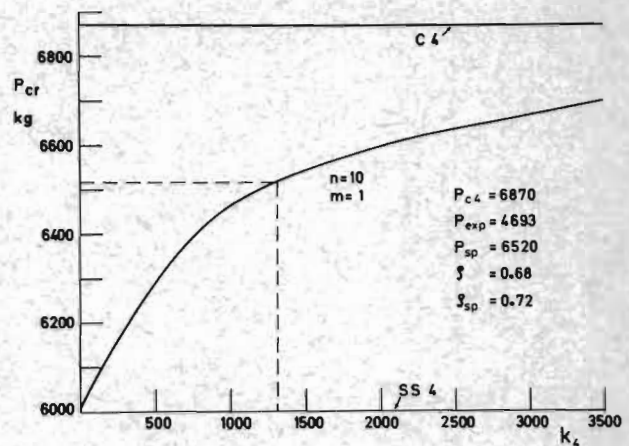


Figure 13. Influence of Elastic Rotational Restraints on the Buckling Loads of Shell RO-31.

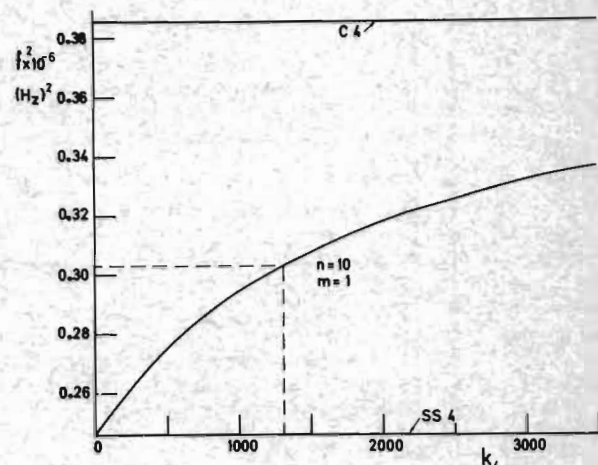


Figure 14. Influence of Elastic Rotational Restraints on the Vibrations of Shell RO-31 ( $P = 1600$  kg.,  $n = 10, m = 1$ ).

restraints on the frequency squared of the model  $n = 12, m = 1$ , at an axial load of 1600 kg. is shown in the same manner in which the influence on buckling load was shown in Fig. 13. This mode of vibration was chosen, because it represents the buckling mode in most of the range of the springs. The behavior is indeed very similar to that shown in Fig. 13. Hence by measuring the natural frequency at a relatively low load, one can estimate the real boundary conditions. For simple supports a similar procedure can be employed by introduction of an axial spring  $k_1$  between SS3 and SS4<sup>28</sup>.

In Fig. 15 the predicted and experimental frequencies squared are plotted versus axial load for

shell RO-31. The experimental results show a clear trend to a certain value of  $k_4$ . In Fig. 16 a

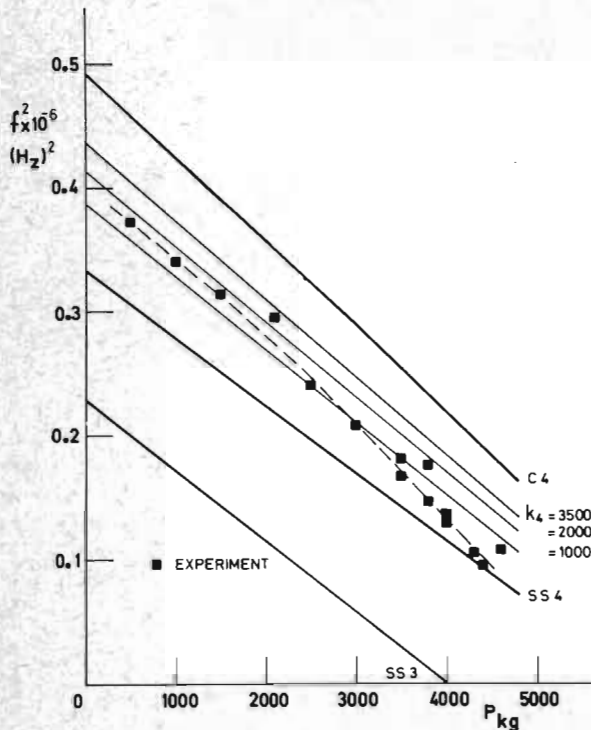


Figure 15. Vibration of a Loaded Stringer-Stiffened Cylindrical Shell for Determination of Boundary Conditions - Frequency Squared Versus Axial Load for Shell RO-31 ( $n = 10$ ,  $m = 1$ ).

A similar plot is shown for the same shell but for another mode  $n = 7$ ,  $m = 1$ . Similar clear trends, even better defined, are observed and the corresponding  $k_4$  value is close to that in Fig. 15. This value  $k_4 = 1300$  is now used in Fig. 13 to estimate the buckling load for the real incomplete clamping yielding  $P_{cr} = 6520$  kg. whereas for C4 conditions  $P_{cr} = 6870$  kg. As the shell buckled at 4693 kg. the "linearity" which was for C4  $\rho_{C4} = 0.68$  is raised to  $\rho_{sp} = 0.72$ .

In Ref. 28 the procedure was applied to four shells. Additional correlations have been carried out, and a total of eighteen simply supported and clamped shells tested may now be summarized. The detailed geometries of all the shells are given in <sup>62</sup>. They cover the ranges:  $(R/h) = 470-510$ ,  $(L/R) = 1.0-2.0$ ,  $(A_1/b_1h) = 0.20-0.80$ ,  $(I_{11}/b_1h^3) = 0.08-4.8$ ,  $-(e_1/h) = 1.5-4.5$ ,  $\eta_{e1} = 0.8-10.4$ . The "linearities"  $\rho_{SS3}$  and  $\rho_{C4}$ , which are plotted in Figs. 3 and 4 (together with those of many other tests), are replotted in Fig. 17 (the open symbols). They are corrected by this correlation with vibrations (as demonstrated for RO-31) to  $\rho_{sp}$ , which is also plotted versus the area ratio in Fig. 17. The experimental scatter in "linearity" which for the same shells was 0.6 - 1.3 without the corrections for real boundary conditions, has been considerably reduced now to 0.6 - 0.9. It should be noted that the low values of  $\rho$  in Fig. 17 relate to clamped shells, as also observed before for

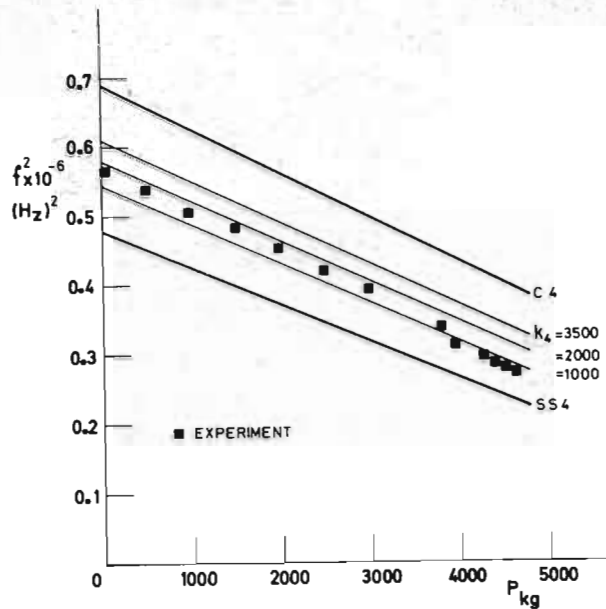


Figure 16. Vibrations of a Loaded Stringer-Stiffened Cylindrical Shell for Determination of Boundary Conditions - Frequency Squared Versus Axial Load for Shell RO-31 ( $n=7$ ,  $m=1$ ).

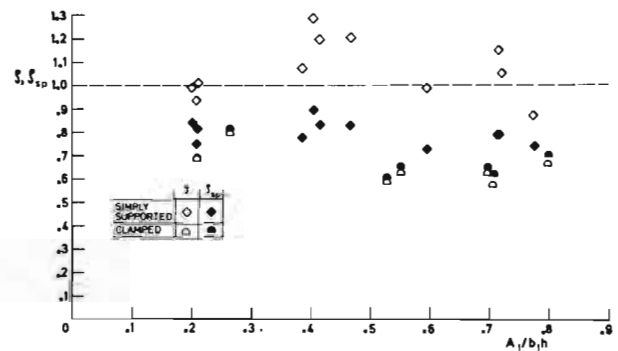


Figure 17. "Linearity" of Simple Supported and Clamped Stringer-Stiffened Shells of RO series<sup>62</sup>, Corrected for Experimentally Determined Boundary Conditions ( $\rho_{sp}$ , the corrected "linearity" is compared to original  $\rho_{SS3}$  and  $\rho_{C4}$  Values).

$\rho_{C4}$  in Fig. 4. This may be due to more severe imperfections introduced by clamping in these tests than in the simple supports used, or perhaps to a difference in imperfection sensitivity for clamped boundaries. Further theoretical and experimental study is required to clarify this observation.

The usefulness of the vibration technique for determination of the real boundary conditions of stringer-stiffened cylindrical shells is therefore twofold: as a tool for a specific shell and as a means to improve empirical design data. AS



mentioned in Section 5, the method includes indirectly also some correlation of initial imperfections. The definition of this correlation requires further study.

### IX. Nondestructive Test Methods for Prediction of Buckling Loads

The prediction of the buckling loads by non-destructive test methods has been attempted by many investigators for different structures. For columns and frames, good results have been obtained by prediction from vibrations (see for example Refs. 91 or 92), but applications of similar vibration techniques to plates and thin shells have not yielded practical methods<sup>91,89</sup>, except one successful application to spherical caps<sup>93</sup>. Other methods of nondestructive testing have recently been developed for panels<sup>94</sup> and for shells<sup>95-97,60</sup> with some success. The results of Ref. 60 are very impressive, though the shells tested there were fairly thick, with  $(R/h) < 200$ . Very good preliminary results were also reported there<sup>60</sup> for a vibration method, which correlates the variation of minimum local dynamic mass at given frequencies with axial load, applied to similar thick shells.

Since the authors believe that correlation with vibration tests is a promising direction of attack, in particular for closely stiffened shells, where the low vibration modes observed in tests are very similar to the buckling modes, such studies were actively pursued<sup>37,28</sup>. Some of the earlier attempts<sup>37</sup> yielded some promising results, but better correlations were obtained with the later tests on shells with heavier stringers<sup>28</sup>. Some preliminary results of these tests were given in Figs. 13 and 14 of Ref. 28. Additional tests have been carried out and the results for a typical clamped shell, RO-31, are shown in Fig. 18. The plot of the square of

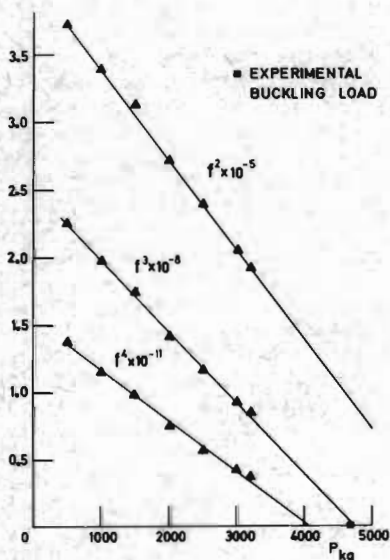


Figure 18. Prediction of Buckling Load from Variation of Measured Frequencies with Axial Load - Shell RO-31.

measured frequencies (which would only be a straight

line for a perfect shell with SS3 boundary conditions) curves down at higher loads (see also Fig.15). Curve fitting of the experimental results of  $f^2$  for P up to about half the theoretical buckling load, was tried but found to be inconvenient. As an alternative, an indirect curve fitting technique was used - straight lines were fitted (least square error fit) to  $f^2$  and to higher powers of the measured frequencies. The extrapolated intersections of these straight lines for  $f^q$  (where  $q = 2,3,4...$  is an empirical exponent representing a form of curve fitting) are possible predicted buckling loads. For RO-31 (as well as for RO-33 and RO-34 studied in Ref. 28) the extrapolations of  $f^3$  and  $f^4$  bracket the experimental buckling loads fairly well, with  $f^4$  yielding conservative predictions. These and similar results for other shells are encouraging, but since the exponents for these semi-empirical straight lines depend on shell geometry and boundary conditions, extensive further studies are needed to develop a reliable non-destructive test method for stiffened cylindrical shells. These studies will also consider the shape of the curves of  $f^2$  for actual imperfect shells, in order to provide a sounder theoretical basis to the empirical exponent  $q$ .

### X. Structural Efficiency

The main motivation for using stiffened shells is their relatively high structural efficiency. Considerable efforts have therefore been devoted in recent years to studies aimed at improving this structural efficiency (see for example Refs. 47,25 or 1). In a paper on recent design criteria<sup>32</sup>, however, ring-stiffened shells under axial compression were stated to be always less efficient than equivalent isotropic shells. This statement was shown in Ref. 1 to be incorrect, but has very recently been restated<sup>61</sup>. In view of its importance to design, the reasoning of Refs. 25 and 1 is therefore briefly reviewed, and then the structural efficiency of stiffened shells is discussed from a slightly different point of view, which even reinforces the earlier conclusions<sup>25,1</sup>.

The structural efficiency of shells, from a design point of view, is evaluated by comparison with equal weight, or "equivalent", unstiffened shells. The thickness of the equivalent shell  $\bar{h}$  is for stringer-stiffened shells

$$\bar{h} = [1 + (A_1/b_1h)]h \quad (3)$$

or for ring-stiffened shells

$$\bar{h} = [1 + A_2/b_2h)]h \quad (4)$$

In the absence of reliable theoretical estimates for unstiffened cylindrical shells under axial compression, one has to rely on empirical formulae<sup>4,5</sup> which show the primary dependence of the buckling coefficient on  $(R/h)$ . Among them there is a very simple formula, proposed by Pflüger<sup>98</sup> for  $(R/h) > 200$ ,

$$(P_B/P_{C1}) = [1 + (R/100h)]^{-1/2} \quad (5)$$

where  $P_{C1} = [3(1-\nu^2)]^{-1/2} \cdot 2\pi E h^2$ , (6) the classical buckling load, that in addition to its simplicity has the additional merit - for the purpose of comparison - of being unconservative for most test data. In Fig. 26 of Ref. 99, Pflüger's formula, Eq. (5), is superimposed on test results obtained by 14

investigators<sup>5</sup> and found to be an upper bound for practically all the shells tested. Hence for the conservative method of Ref. 25,  $\bar{P}_B$  from Eq. (5) is a suitable standard for comparison - since the structural efficiency obtained related to it is smaller than the actual efficiency of the stiffened shell.

If Eq. (5) is now employed for the equivalent unstiffened shell defined by Eqs. (3) or (5) its buckling load is

$$\begin{aligned}\bar{P}_B &= \bar{P}_{cl} [1 + (R/100\bar{h})]^{-1/2} \\ &= P_{cl} (\bar{h}/h)^2 [1 + (R/100\bar{h})]^{-1/2}\end{aligned}\quad (7)$$

Now since for ring-stiffened cylindrical shells with outside rings the general instability load can be computed with a simple formula (Eq. (3) of Ref. 1, or Eq. (9) of Ref. 25) a conservative efficiency for these shells is given by

$$\begin{aligned}\eta &= \rho (P_{stiff}/\bar{P}_B) = \\ &= \rho [(\bar{h}/h) = (R/100\bar{h})]^{1/2} (\bar{h}/h)^{-2}\end{aligned}\quad (8)$$

where  $\rho$  is the "linearity".

With Eqs. (8) and (4) design curves can readily be drawn that show  $\eta$  versus  $(R/h)$  for various values of  $\rho$  and  $(A_2/b_2h)$ . In Fig. 19 (reproduced from Ref. 25), a typical set of such

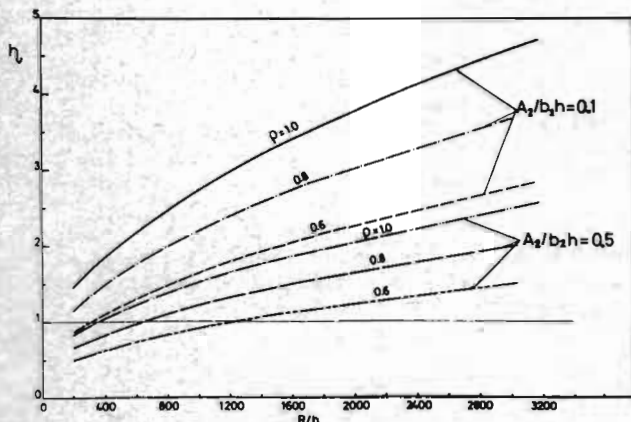


Figure 19. Structural Efficiency of Axially Compressed Ring-Stiffened Cylindrical Shells (Outside Rings - Reproduced from Ref. 25).

curves is presented. It is immediately seen that even when  $\rho$  is only 0.6, weak ring-stiffening is very efficient for thin shells (large  $R/h$ ); or, in other words, thin shells with many closely spaced rings (to prevent local buckling) and external rings of small cross-sectional area carry axial compression very efficiently.

For shells stiffened by internal rings, or stringers, the general instability load has to be computed from slightly more elaborate expressions than the simple formula referred to. The conservative efficiency may then be written for both cases as

$$\begin{aligned}\eta &= \rho (P_{stiff}/\bar{P}_B) = \\ &= \rho P_{stiff} [3(1-\nu^2)]^{1/2} [2\pi E h^2]^{-1} [(\bar{h}/h) + (R/100\bar{h})]^{1/2} \\ &\quad \cdot (\bar{h}/h)^{2.5}\end{aligned}\quad (9)$$

Design curves have been computed with the aid of Eqs. (9) and (3) for stringer-stiffened cylindrical shells under axial compression (see for example Fig. 14 of Ref. 25) and the stiffening has generally been found to be more efficient for lighter stringers and thinner shells. One should recall that  $\eta$  computed for stiffened shells by this method is smaller than their actual efficiency. However, even with this conservative criterion, the ring and stringer-stiffened shells tested in the experimental program discussed<sup>5,34</sup> show fairly high efficiencies (see Fig. 15 of Ref. 25 and Fig. 14 of Ref. 1) and demonstrate the advantages of stiffening.

The structural efficiency of stiffened shells can be studied in a slightly different manner. For perfect shells, the theoretical efficiency would be simply the ratio of the general instability load to that of the classical buckling load of the equivalent weight isotropic shell. For real shells the predicted loads have to be multiplied by the expected "linearities". Hence

$$\eta = [P_{stiff} \cdot \rho_{stiffened}/\bar{P}_{cl} \cdot \rho_{eq}] \quad (10)$$

If  $\gamma$  is defined as the ratio of "linearities" (or knock-down factors)

$$\gamma = \rho_{stiffened}/\rho_{eq} \quad (11)$$

the efficiency becomes

$$\eta = \gamma [P_{stiff}/\bar{P}_{cl}] \quad (12)$$

where only  $\gamma$  is empirical, and the remaining quantities are simple theoretical values predicted by linear theory. The designer uses curves that present  $\eta$  for various values of  $\gamma$  ( $\gamma = 1, 1.1, 1.2$  etc), and chooses the appropriate  $\gamma$  for this structure. In this method it has been tacitly assumed that for isotropic cylindrical shells under axial compression the influence of the boundary conditions is negligible<sup>20</sup> (the "weak shear" SS1 and SS2 B.C.'s are not considered here). On the other hand, for stringer-stiffened shells the boundary conditions have been shown to be very important and have, therefore, to be taken into account.

The "linearity"  $\rho$  is taken from test data. In order to remain conservative, the lower bound of the test results is used. For simple supports,  $\rho_{SS3} \geq 0.67$  in Fig. 3, if very weakly stiffened shells are excluded by restricting  $(A_2/b_2h) \geq 0.15$  and the shells of Ref. 65 are also excluded due to occurrence of local buckling, as discussed in Section 2. Since the AS shells and RO shells are known to have B.C.'s between SS3 and SS4, their "linearity" referred to SS4 B.C.'s has also been plotted (Fig. 20). A lower value than in Fig. 3 is obviously obtained for the lower bound in Fig. 20 yielding  $\rho_{SS4} \geq 0.57$ . For the clamped shells in Fig. 4 the lower bound is  $\rho_{C4} \geq 0.58$ . The "linearity" of the isotropic shells is taken from results for six unstiffened shells tested in the RO-series, which were manufactured by the same process as the stiffened shells. For a range of  $(R/h) \approx 400$  to 520 in these six shells, the lower bound of

linearity is  $\rho \approx 0.45$ . To take account of the decrease in  $(R/h)$  in the equivalent shells, a value  $\rho_{eq} = 0.50$  is taken for the comparisons. This

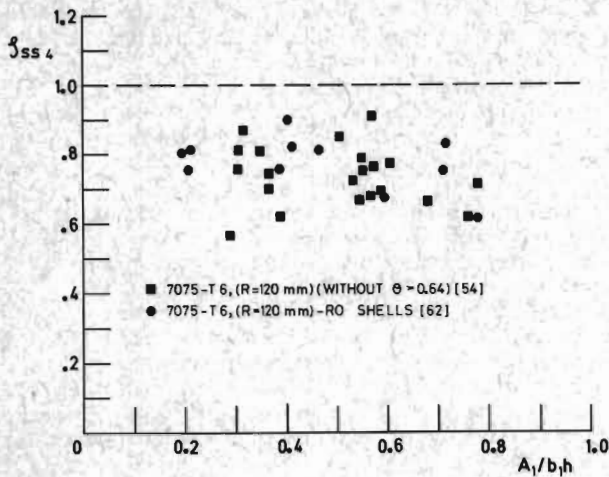


Figure 20. "Linearity" of Simply Supported Stringer-Stiffened Cylindrical Shells under Axial Compression, Referred to SS4 Boundary Conditions, as Function of Stringer Area Parameter. (AS Shells<sup>54</sup> and RO Shells<sup>62</sup>).

corresponds to an extrapolation for the equivalent shells in the case of  $(A_1/b_1h) \approx 0.3$ . This value of  $\rho$  is, however, considerably above the lower bound of Weingarten and Seide<sup>5</sup> even for  $(R/h) = 240$ , that would correspond to the equivalent shell in the case of  $(A_1/b_1h) = 1.0$ . With this value of  $\rho$  for isotropic shells,  $\gamma = 1.34$  for SS3,  $\gamma = 1.14$  for SS4 and  $\gamma = 1.16$  for C4. In Fig. 21,  $\eta$  for SS3 is plotted for  $\gamma = 1.4$  and  $(L/R) = 1$ . Note that shell geometries of Table 1 are used in these parametric studies. A minimum  $\eta = 1.1$  occurs when  $(A_1/b_1h) \approx 0.3$ . Similar calculations for SS3 show that shorter shells have higher efficiencies, whereas  $\eta$  may even drop below 1.0 for long shells. The other curves in Fig. 21 relate to SS4 B.C.'s. Again the increase in  $\eta$  for shorter shells is apparent, but in particular one may note the relative high value of  $(A_1/b_1h) \approx 0.8$  for maximum  $\eta$ . With  $\gamma = 1.2$ , efficiencies of 1.2 for  $(L/R) = 2$  and 1.4 for  $(L/R) = 1$  occur for practical area ratios. In Fig. 22 the efficiencies related to C4 are plotted. Here both  $(L/R)$  and  $(A_1/b_1h)$  have a stronger influence on  $\eta$  than in the other cases, though the trends are similar to those observed for SS4 boundaries. For  $\gamma = 1.2$  (which is close to the 1.16 computed from the test data) the stringer-stiffened shells are always efficient, in the range of geometries studied, and for short shells,  $(L/R) = 1$ , very high values of  $\eta$  are possible for heavy stiffening. In that case, however, due to the relatively high buckling stresses, inelastic effects may come into play, as discussed in Section 6.

The results presented in Figs. 21 and 22 demonstrate again that stringer-stiffened shells are usually much superior than equivalent isotropic ones. It may be recalled that the parametric studies have been carried out with the geometries

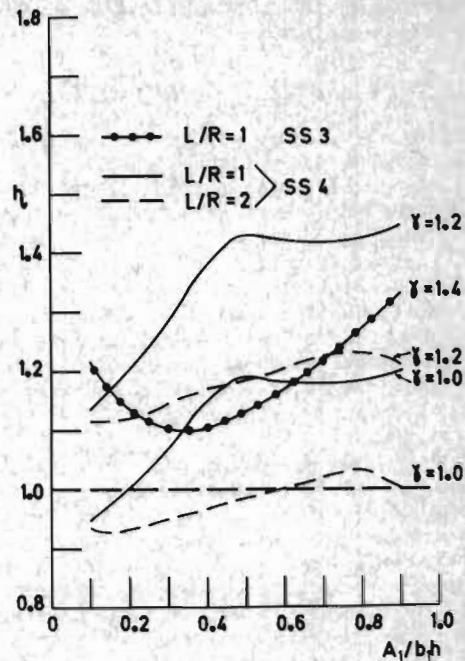


Figure 21. Structural Efficiency of Axially Compressed Simply Supported Stringer-Stiffened Cylindrical Shells as Function of Stringers Area Parameter, for Different "Linearity" ratios  $\gamma$ . (Stringer Geometry is Given in Table 1).

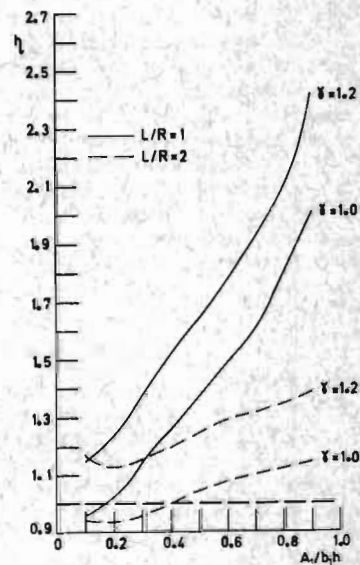


Figure 22. Structural Efficiency of Axially Compressed Clamped Stringer-Stiffened Cylindrical Shells as Function of Stringer Area Parameter, for Different "Linearity" ratios  $\gamma$ . (Stringer Geometry is Given in Table 1).



of Table 1, which represent the relatively inefficient stiffeners of the test cylinders. For more efficient stringers used in practical designs, the efficiencies will be considerably higher. Any doubt about the intrinsically high structural efficiency of stiffened shells should therefore by now have been dispelled completely.

#### XI. Design Criteria and Future Trends

In spite of vigorous efforts to develop rational design criteria for stiffened shells based on axisymmetric imperfection theory<sup>31,32</sup>, these have not yet reached the stage of general practical applicability. Only interim criteria<sup>31</sup> or criteria based on very restrictive assumptions<sup>32</sup> have been proposed, and the designer still has to rely essentially on semi-empirical methods based on correlation of linear, or nonlinear, theories with typical test data. Improvements in these correlations, such as those discussed in the paper, will therefore yield improved design criteria.

The design process can usually be divided into two phases: (a) the definition of the optimal configuration, and (b) the evaluation of the chosen configuration and its detail design. For synthesis of the optimization phase, the computational procedures and programs must be simple. After consideration of the many effects and the parameters which determine their magnitude, one is led back to linear theory as the most suitable tool for the optimization phase of design of stiffened shells. The "applicability" of linear theory (with design factors  $\rho \approx 0.7-0.8$  in general) as a first approximation has been reconfirmed by additional tests, and the general bounds of applicability under axial compression (summarized in Ref. 1) have been verified. Due to the pronounced influence of boundary conditions on stringer-stiffened shells, their definition by the designer is important and correlation with boundary conditions other than the classical SS3 may be preferable (as has been proposed in Section 10). For shells made of materials with early nonlinear behavior, the use of maximum strength analyses<sup>23</sup> may be advisable even in the optimization process prior to the definition of the configuration. The method of load application may also have to be considered at this early stage on account of possible load eccentricity effects. In general, however, for this phase in design, the linear theories (with appropriate correlation factors) seem satisfactory. The relevant detailed design criteria are summarized in the Conclusions.

For the second phase of design, more sophisticated methods of analysis should be employed (see for example Ref. 100). Boundary conditions must be accurately determined, and correlation with vibrations of typical test shells may assist the designer in this task. The effects of imperfections have to be taken into account, and it is important that real imperfections, typical of the shell design, are considered. Initial imperfections of typical test articles have therefore to be measured. With measured initial imperfections, representative of the actual shell, and consideration of realistic boundary conditions, fairly accurate predictions may be expected with recently developed methods<sup>22</sup>. Inelastic effects and material nonlinearities should also be considered, in particular their possible interaction with initial imperfections. Possible

load eccentricity effects have also to be taken into account in both analysis and detail design.

Some of the above recommendations extend beyond the present state of the art. The current trends of research and development in this field seem to focus on measurement of initial imperfections, and their growth under load, for laboratory and full-scale shells and their correlation with manufacturing processes and multi-mode theoretical predictions, on the one hand, and development of reliable non-destructive test methods, on the other hand. If one extrapolated these trends, the following design process of stiffened shells could be envisaged in the near future:

The definition of the optimal configuration is carried out with automated design programs, using linear theory in the structural analysis parts related to proper elastic boundary conditions (estimated from earlier test experience) and correlated with correlation factors  $\rho$  obtained from data accumulated from many experiments run in different laboratories. The correlation factors are also modified by the planned production process whose typical initial imperfection properties are available from accumulated test data.

In the evaluation and final design phase that follows, the large structural analysis programs use typical detailed predicted initial imperfection patterns from the same test data to predict the buckling loads, while ensuring that realistic boundary conditions are assumed and that inelastic effects and material nonlinearities are also properly accounted for. A prototype is tested in a comprehensive manner - including measurement of initial imperfections, vibrations to verify or modify the assumed boundary conditions, and a pilot non-destructive test method which is being recalibrated in this test (which is carried to destruction). The imperfection and boundary condition measurements of the test are used to verify or modify the earlier predictions. Then, as the shells come off the production line, the initial imperfections of each one are measured as part of the quality control process, and the relevant buckling loads recomputed accordingly, qualifying each shell. As part of the same control process, some of the shells are mounted in conditions simulating their actual boundary conditions and tested non-destructively to verify the buckling load the shell will carry in these conditions. A rather elaborate process, but essential to ensure the accuracy required to make really significant structural weight reductions feasible.

#### XII. Conclusions

The following conclusions can be drawn from the results discussed:

- (1) Linear theory is applicable to closely stiffened ring-and stringer-stiffened cylindrical shells as a first approximation, and is a suitable tool for optimization studies and definition of design configurations of stiffened shells.

Extensive experimental evidence confirms this for the most severe loading case - axial compression, and results for other loading cases strongly support the conclusion. This "applicability" includes small design factors

implying reductions of 20-30%, but the customary large "knock-down factors" are absent even for weak stiffening, provided local buckling is excluded.

- (2) The "linearity  $\rho = P_{exp}/P_{cr}$ " depends on stiffener and shell geometry, primarily on the stiffener area ratio, and on the boundary conditions. For the preliminary design phase and optimization studies, the "linearity" in the case of axial compression may be taken, when referred to SS3 B.C.'s as  $\rho_{SS3} \approx 0.7$  for  $0.15 \leq (A_1/b_1h) \leq 0.5$  and  $\rho_{SS3} \approx 0.8$  for  $(A_1/b_1h) > 0.5$  for stringers, with similar slightly higher values of  $\rho$  for rings.

Correlation with linear theory for other boundary conditions may be preferable, and then  $\rho_{SS4} \approx 0.7$  and  $\rho_{C4} \approx 0.65$  are suitable values for the preliminary design phase.

Note that these values of  $\rho$  are not the lower bounds observed in tests, since this would make the optimization unduly conservative. The lower bounds should however be used in the final design phases. The appropriate values of  $\rho$  are then, again for axial compression,  $\rho_{SS3} > 0.67$ ,  $\rho_{SS4} > 0.57$  and  $\rho_{C4} > 0.58$ . For other loadings higher values of  $\rho$  are observed for the respective boundary conditions.

- (3) Smeared stiffener theory is a convenient method of analysis for general instability of integrally stiffened shells. Eccentricity of stiffeners strongly influences the buckling load, and this eccentricity effect depends on stiffener and shell geometry, as well as on the type of loading. Discreteness of stiffeners can usually be ignored.
- (4) Boundary conditions are of prime importance in stringer-stiffened shells, whereas for ring-stiffened shells their influence is much smaller.

Axial restraints, and rotational restraints, strongly affect the buckling loads and free vibrations of stringer-stiffened shells. This effect depends on shell and stiffener geometry. This analogical behavior can be utilized as a tool for the better definition of the actual boundary conditions, which will lead to more accurate predictions of buckling loads and will significantly reduce experimental scatter.

- (5) For externally stringer-stiffened cylindrical shells under axial compression, in the range of geometries studied, nonlinear prebuckling deformations have a relatively small effect on the buckling loads and vibrations, except for very short shells where consideration of prebuckling deformation yields significantly higher buckling loads.
- (6) Eccentricity of loading is controlled by the designer, but he should remember that it has a very significant effect on the buckling load, behavior and vibrations of stringer-stiffened shells.

- (7) The influence of initial imperfections, though less pronounced for closely stiffened shells, is an important degrading factor. Initial imperfections also affect vibrations significantly. The importance of taking realistic imperfections into account should be borne in mind as well as the resultant need for extensive imperfection measurements.
- (8) Inelastic effects and material nonlinearity may influence buckling loads even at fairly large  $(R/h)$  ratios, and their degrading effect may be of similar magnitude to that caused by initial imperfections. These effects should therefore not be ignored.
- (9) Preliminary results indicate that correlation with vibration tests, at axial loads much below the buckling loads, may yield the basis for a non-destructive test method for prediction of buckling load of imperfect closely stiffened shells.
- (10) Structural efficiency studies, that consider test results for stiffened and unstiffened cylindrical shells, emphasize the superiority of stiffened shells over the corresponding equivalent weight isotropic ones. The structural efficiency of stringer-stiffened shells is improved with increase in rigidity of the boundary conditions.

#### Acknowledgement

The authors wish to thank Mr. H. Abramowitz, Mr. A. Greenwald and Mr. A. Klausner for their dedicated assistance in the experimental work and data processing. They also wish to thank Mr. S. Nachmani, Mrs. L. Spector, Mrs. J. Kessler, Mrs. S. Shalem for their assistance and Mrs. D. Zirkin for the typing of the manuscript, and Mrs. D. Rosen for preparation of the Figures.

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mization phase of design. On the other hand, introduction of an arbitrary safety margin to account for possible modal interactions, could be justified only on the basis of extensive calculations or tests of typical shells, and therefore does not appear advisable before such data is available.

The low value of "linearity" observed in the single test mentioned by Prof. Rehfield, may be due to some effects peculiar to that test. Considerably higher values, as presented in the paper, were obtained in all the tests of closely stiffened shells performed by many different investigators (see for example Figs.3 and 4 of the paper).

Hence for the optimization phase in design, the linear theories (with appropriate correlation factors) appear to be satisfactory tools, provided their approximate nature is appreciated. In the second design phase - evaluation of the chosen configuration - the more sophisticated methods of analysis should consider modal interaction as well, but focus their attention on the effect of real imperfections, as measured on typical stiffened shells.

#### D I S C U S S I O N

L.W. Rehfield (School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, U.S.A): I should like to comment on Prof. Singer's remarks on optimization using linear theory. While I cannot at this time offer an alternative, such an approach may well lead to a mediocre design that is far from optimal. This is due to possible nonlinear interaction of near-simultaneous failure modes such as local skin buckling, stiffener crippling, panel instability and other competing general instability modes with the mode predicted by linear theory. This matter is discussed in Prof. van der Neut's paper for a simpler structure. Such a possibility has been accounted for approximately in practical design situations by employing the design approach of Ref. 33 with a safety margin of approximately ten percent against simultaneous buckling in local modes. This was done as a practical expedient in order to produce general instability in a near optimal shell for the experimental program at the Georgia Institute of Technology. The first shell designed in this way buckled at a "linearity" of approximately 0.4, in a mode that is dominated by the one predicted by linear theory, a value well below the data presented by Prof. Singer. Data reduction is not complete, so an assessment of effects such as boundary restraint, has not yet been made. While one cannot generalize on the basis of a single test, a word of caution on optimization for buckling resistance on the basis of linear theory is, nevertheless, appropriate.

J. Singer and A. Rosen: The importance of the possible nonlinear interaction of near-simultaneous failure modes, mentioned by Prof. Rehfield, as a likely degrading factor of an optimal structure predicted by linear theory should certainly not be overlooked. A theory for stiffened shells that analyses modal interaction, would however at present be too complicated for inclusion in the opti-